

# Wermer type sets and extension of CR functions

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**Abstract.** For each  $n \geq 2$  we construct an unbounded closed pseudoconcave complete pluripolar set  $\mathcal{E}$  in  $\mathbb{C}^n$  which contains no analytic variety of positive dimension (we call it a *Wermer type set*). We also construct an unbounded strictly pseudoconvex domain  $\Omega$  in  $\mathbb{C}^n$  and a smooth *CR* function  $f$  on  $\partial\Omega$  which has a single-valued holomorphic extension exactly to the set  $\overline{\Omega} \setminus \mathcal{E}$ .

## 1. Introduction

In this paper we are dealing with the extension problem of *CR* functions defined on the boundary  $\partial\Omega$  of an unbounded domain  $\Omega$  in  $\mathbb{C}^n$ ,  $n \geq 2$ . When  $\Omega$  is bounded with a connected smooth boundary (no hypothesis of pseudoconvexity) holomorphic extension of *CR* functions to the whole of  $\Omega$  is granted by the classical result of Bochner (see, for example, Theorem 2.3.2' in [H]). In particular, if  $\Omega$  is a domain of holomorphy, the envelope of holomorphy  $E(\partial\Omega)$  of  $\partial\Omega$  (i.e. the envelope of  $\partial\Omega$  with respect to the algebra of *CR* functions on  $\partial\Omega$  (for details see, for example, [J], [MP], [St])) coincides with  $\overline{\Omega}$ . For unbounded domains such an extension result is not longer true in general, even for strictly pseudoconvex domains, as shown by the following example.

**Example.** Let  $f$  be an entire function in  $\mathbb{C}^2$  and

$$\Omega := \{z \in \mathbb{C}^2 : \log|f(z)| + C_1\|z\|^2 < C_2\}$$

where  $C_1$  and  $C_2$  are constants and  $C_1 > 0$ . For almost all constants  $C_2$ ,  $\Omega$  is an unbounded strictly pseudoconvex domain with smooth boundary in  $\mathbb{C}^2$  containing

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the divisor  $\{f = 0\}$ . We are going to show that  $E(\partial\Omega)$  is one-sheeted, contained in  $\Omega$  and

$$\overline{\Omega} \setminus E(\partial\Omega) = \{z \in \mathbb{C}^2 : f(z) = 0\}.$$

Fix an exhaustion  $V_1 \subset\subset V_2 \subset\subset \dots \subset\subset \partial\Omega$  of  $\partial\Omega$  by relatively compact subsets. Intersecting  $\Omega$  by balls  $B^2(0, R_k) \subset \mathbb{C}^2$  centered at the origin of radius  $R_k$  in such a way that  $V_k \subset\subset \partial\Omega \cap B^2(0, R_k)$  and then smoothing the edges as in [To] we can find strictly pseudoconvex bounded domains  $\Omega_k$  in  $\mathbb{C}^2$  such that  $V_k \subset \partial\Omega_k \cap \partial\Omega$  for every  $k \in \mathbb{N}$ . Let  $\Gamma_k := \partial\Omega_k \setminus V_k$ . Then, in view of Theorem A from [J], one has

$$E(V_k) = E(\partial\Omega_k \setminus \Gamma_k) = \overline{\Omega}_k \setminus \widehat{(\Gamma_k)}_{\mathcal{A}(\Omega_k)} \subset \mathbb{C}^2,$$

where  $\widehat{(\Gamma_k)}_{\mathcal{A}(\Omega_k)}$  is the  $\mathcal{A}(\Omega_k)$ -hull of  $\Gamma_k$ , i.e. the hull of  $\Gamma_k$  with respect to the algebra of holomorphic functions on  $\Omega_k$  which are continuous up to the boundary. It follows that  $E(\partial\Omega) := \bigcup_{k=1}^{\infty} E(V_k)$  is one-sheeted. We just have to show that  $\overline{\Omega} \setminus E(\partial\Omega)$  is the divisor  $\{f = 0\}$ . Since the  $CR$  function  $1/f$  on  $\partial\Omega$  does not extend to  $\{f = 0\}$ , it follows that  $\{f = 0\} \subset \overline{\Omega} \setminus E(\partial\Omega)$ . Conversely, filling  $\overline{\Omega} \setminus \{f = 0\}$  by the following family of holomorphic curves,  $\gamma_t = \{z \in \overline{\Omega} \setminus \{f = 0\} : f(z) = e^{i\theta}t\}$ , where  $\theta \in [0, 2\pi]$ ,  $t \in \mathbb{R}$ , and using the Kontinuitätssatz, it turns out that  $\overline{\Omega} \setminus \{f = 0\} \subset E(\partial\Omega)$ .

In this context we have to mention Trépreau's Theorem [Tr] stating that, given a point  $z$  in a smooth hypersurface  $M \subset \mathbb{C}^n$ , the homomorphism

$$\mathcal{O}_z \rightarrow \varinjlim_{U \ni z} \mathcal{O}(U \setminus M)$$

is onto if and only if no germ of a complex hypersurface passing through  $z$  is contained in  $M$ . We also recall Chirka's generalization [C] of Trépreau's result (in the case  $n = 1$  this generalization can also be obtained from the earlier work [Sh]): let  $\Gamma \subset \mathbb{C}^{n+1}$  be a continuous graph over a convex domain  $D \subset \mathbb{C}^n \times \mathbb{R}$  and  $z \in \Gamma$  be a point such that none of the connected components of  $(D \times \mathbb{R}) \setminus \Gamma$  is extendable holomorphically to  $z$ . Then,  $z$  is contained in an  $n$ -dimensional holomorphic graph lying on and closed in  $\Gamma$ .

A natural question arises: let  $\Omega$  be an unbounded strictly pseudoconvex domain in  $\mathbb{C}^n$ ,  $n \geq 2$ , such that  $E(\partial\Omega)$  is one-sheeted and  $\overline{\Omega} \setminus E(\partial\Omega) \neq \emptyset$ ; does  $\overline{\Omega} \setminus E(\partial\Omega)$  possess an analytic structure? In this paper we prove that the answer to this question is negative. Precisely, we prove the following two theorems.

**Theorem 1.** *For each  $n \in \mathbb{N}$ ,  $n \geq 2$ , there exist a closed set  $\mathcal{E} \subset \mathbb{C}^n$  which contains no analytic variety of positive dimension and a plurisubharmonic function  $\varphi : \mathbb{C}^n \rightarrow [-\infty, +\infty)$  such that:*

- 1)  $\mathcal{E} = \{z \in \mathbb{C}^n : \varphi(z) = -\infty\}$ .
- 2) The function  $\varphi$  is pluriharmonic on  $\mathbb{C}^n \setminus \mathcal{E}$ .

- 3) The domain  $\mathbb{C}^n \setminus \mathcal{E}$  is pseudoconvex.
- 4) For every  $R > 0$  one has  $\widehat{\partial B^n(0, R) \cap \mathcal{E}} = \overline{B^n(0, R)} \cap \mathcal{E}$ , where  $B^n(0, R) \subset \mathbb{C}^n$  is the ball of radius  $R$  centered at the origin and  $\widehat{\partial B^n(0, R) \cap \mathcal{E}}$  denotes the polynomial hull of the set  $\partial B^n(0, R) \cap \mathcal{E}$ .

**Theorem 2.** For each  $n \in \mathbb{N}$ ,  $n \geq 2$ , there exist an unbounded strictly pseudoconvex domain  $\Omega$  in  $\mathbb{C}^n$ , a closed subset  $\mathcal{E}$  of  $\mathbb{C}^n$  and a  $CR$  function  $f$  on  $\partial\Omega$  such that:

- 1)  $\mathcal{E} \subset \Omega$  and it contains no analytic variety of positive dimension.
- 2)  $f$  has a single-valued holomorphic extension exactly to  $\Omega \setminus \mathcal{E}$ .
- 3) The envelope of holomorphy  $E(\partial\Omega)$  of the set  $\partial\Omega$  is one-sheeted and  $E(\partial\Omega) = \overline{\Omega} \setminus \mathcal{E}$ .

The set  $\mathcal{E}$  is obtained as a limit in the Hausdorff metric of a sequence  $\{E_\nu\}$  of algebraic hypersurfaces of  $\mathbb{C}^n = \mathbb{C}_z^{n-1} \times \mathbb{C}_w$  such that the union of the corresponding sets of ramification points with respect to the projection  $\mathbb{C}^n \rightarrow \mathbb{C}_z^{n-1}$  is an everywhere dense subset of  $\mathbb{C}_z^{n-1}$ . For  $n = 2$  this idea goes back to Wermer in [W], where an example of a compact set  $K$  in  $\mathbb{C}^2$  with nontrivial polynomial hull  $\widehat{K}$  such that  $\widehat{K} \setminus K$  has no analytic structure is given. Wermer's construction was then further exploited and developed in a series of articles [A], [D], [DS], [EM], [Le], [Sl]. Note also that, first, our construction of  $\mathcal{E}$  is slightly different from Wermer's one (the main idea being the same) and, secondly, that, in the general case  $n > 2$ , the situation is substantially more difficult from the technical point of view than that considered by Wermer.

Finally, let us mention a result due to Lupaccolu [Lu] about extendability of  $CR$  functions defined on the boundary of an unbounded strictly pseudoconvex domain  $\Omega$ : suppose that there exists a divisor which does not meet the domain  $\Omega$ . Then  $E(\partial\Omega) = \overline{\Omega}$ , namely any  $CR$  function on the boundary extends inside the domain.

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## 2. Construction of an unbounded Wermer type set in $\mathbb{C}^n$

Let  $(z, w) = (z_1, \dots, z_{n-1}, w)$  denote the coordinates in  $\mathbb{C}^n$  and for each  $\nu \in \mathbb{N}$  let  $\mathbb{N}_\nu := \{1, 2, \dots, \nu\}$ . For each  $p \in \mathbb{N}_{n-1}$  fix an everywhere dense subset  $\{a_l^p\}_{l=1}^\infty$

of  $\mathbb{C}$  such that  $a_l^p \neq a_{l'}^p$  if  $l \neq l'$ . Further fix a bijection  $\Phi := ([\cdot], \phi) : \mathbb{N} \rightarrow \mathbb{N}_{n-1} \times \mathbb{N}$  and define a sequence  $\{a_l\}_{l=1}^\infty$  in  $\mathbb{C}$  by letting  $a_l := a_{\phi(l)}^{[l]}$ . Moreover let  $\{\varepsilon_l\}_{l=1}^\infty$  be a decreasing sequence of positive numbers converging to zero that we consider to be fixed, but that will be further specified later on. Then for every  $\nu \in \mathbb{N}$  we define  $g_\nu$  to be the algebraic function

$$g_\nu(z) := \sum_{l=1}^\nu \varepsilon_l \sqrt{z_{[l]} - a_l}$$

and let

$$E_\nu := \{(z, w) \in \mathbb{C}^n : w = g_\nu(z)\}.$$

By definition  $g_\nu$  is a multi-valued function that takes  $2^\nu$  values at each point  $z \in \mathbb{C}^{n-1}$  (counted with multiplicities). Therefore we can always choose single-valued functions  $w_1^{(\nu)}, \dots, w_{2^\nu}^{(\nu)}$  on  $\mathbb{C}^{n-1}$  such that

$$g_\nu(z) = \{w_j^{(\nu)}(z) : j = 1, \dots, 2^\nu\}$$

for all  $z \in \mathbb{C}^{n-1}$ . Note that these functions are not continuous and that they are not uniquely determined, even though the set  $g_\nu(z)$  is well-defined for each  $z \in \mathbb{C}^{n-1}$ . Indeed we may freely change the numeration of the values  $w_1^{(\nu)}(z), \dots, w_{2^\nu}^{(\nu)}(z)$  for each  $z \in \mathbb{C}^{n-1}$ .

Define for each  $\nu \in \mathbb{N}$  a function  $P_\nu : \mathbb{C}^n \rightarrow \mathbb{C}$  as

$$P_\nu(z, w) := (w - w_1^{(\nu)}(z)) \cdots (w - w_{2^\nu}^{(\nu)}(z)).$$

**Lemma 1.** *The sequence  $\{P_\nu\}_{\nu=1}^\infty$  consists of holomorphic polynomials on  $\mathbb{C}^n$  and has the following properties:*

1.  $E_\nu = \{(z, w) \in \mathbb{C}^n : P_\nu(z, w) = 0\}$ .
2.  $P_{\nu+1} \rightarrow P_\nu^2$  uniformly on compact subsets of  $\mathbb{C}^n$  as  $\varepsilon_{\nu+1} \rightarrow 0$ .

**Proof.** First note that if for each  $p \in \mathbb{N}_{n-1}$  we let  $U_p$  be an open convex subset of  $\mathbb{C}$  not meeting  $A_\nu^p := \{a_l : l \in \mathbb{N}_\nu, [l] = p\}$ , then after possibly renumbering the values  $w_j^{(\nu)}(z)$  for  $z \in U := U_1 \times \cdots \times U_{n-1}$  we can always assume the functions  $w_1^{(\nu)}, \dots, w_{2^\nu}^{(\nu)}$  to be holomorphic on  $U$ . Since the value  $P_\nu(z, w)$  is independent of the numeration of the  $w_j^{(\nu)}(z)$ , this shows that  $P_\nu$  is a holomorphic function outside the set  $\mathcal{A}_\nu := \{(z, w) \in \mathbb{C}^n : z_p \in A_\nu^p \text{ for some } p \in \mathbb{N}_{n-1}\}$ . Observing that  $P_\nu$  is locally bounded near each point of  $\mathcal{A}_\nu$  and applying Riemann's removable singularities theorem we conclude that  $P_\nu$  is actually holomorphic in the whole of  $\mathbb{C}^n$ . Then estimating  $|P_\nu|$  outside some ball  $B^n(0, R) \subset \mathbb{C}^n$  from above by a suitable scalar multiple of  $|w^{2^\nu}| + \sum_{p=1}^{n-1} |z_p^{2^{\nu-1}}|$  one can easily see that  $P_\nu$  is in fact a holomorphic polynomial. To prove the second part of the lemma we observe that  $P_{\nu+1}(z, w)$  is

in fact the product of the  $2^\nu$  factors  $((w - w_j^{(\nu)}(z))^2 - \varepsilon_{\nu+1}^2(z_{[\nu+1]} - a_{\nu+1}))$ ,  $j \in \mathbb{N}_{2^\nu}$ , and hence equals

$$\sum_{p=0}^{2^\nu} (-1)^p \left[ (\varepsilon_{\nu+1}^2(z_{[\nu+1]} - a_{\nu+1}))^{2^\nu - p} \cdot \sum_{1 \leq j_1 < \dots < j_p \leq 2^\nu} (w - w_{j_1}^{(\nu)}(z))^2 \cdots (w - w_{j_p}^{(\nu)}(z))^2 \right].$$

Note that for  $p = 2^\nu$  the inner sum equals  $P_\nu^2(z, w)$ . Since  $w_1^{(\nu)}, \dots, w_{2^\nu}^{(\nu)}$  are independent of  $\varepsilon_{\nu+1}$  and bounded on compact subsets of  $\mathbb{C}^{n-1}$ , we conclude that  $P_{\nu+1} \rightarrow P_\nu^2$  uniformly on compact subsets as  $\varepsilon_{\nu+1} \rightarrow 0$ .  $\square$

**Remark.** A more careful consideration shows that one has the following explicit formula for  $P_\nu$ ,

$$P_\nu(z, w) = \sum_{d=0}^{2^{\nu-1}} (-1)^d \left( \sum_{l=1}^{\nu} \varepsilon_l^2(z_{[l]} - a_l) \right)^d w^{2^\nu - 2d}.$$

**Lemma 2.** Let  $\{\varepsilon_l\}$  be chosen in such a way that  $\varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^l$  on  $B^{n-1}(0, l) \subset \mathbb{C}_z^{n-1}$  for every  $l \in \mathbb{N}$ . Then the following assertions hold true:

- 1) For every  $R > 0$  and  $\nu, \mu \in \mathbb{N}$ ,  $\nu \geq R$ , the Hausdorff distance between  $E_\nu \cap \overline{B}^n(0, R)$  and  $E_{\nu+\mu} \cap \overline{B}^n(0, R)$  is less than  $1/2^\nu$ . In particular the sequence  $\{E_\nu \cap \overline{B}^n(0, R)\}_{\nu=1}^\infty$  converges in the Hausdorff metric to a closed set  $\mathcal{E}_{(R)} \subset \overline{B}^n(0, R)$ .
- 2) The union  $\mathcal{E} := \bigcup_{R>0} \mathcal{E}_{(R)}$  of all  $\mathcal{E}_{(R)}$  is a nonempty closed unbounded subset of  $\mathbb{C}^n$  and a point  $(z, w) \in \mathbb{C}^n$  lies in  $\mathcal{E}$  if and only if there exists a sequence of complex numbers  $w_\nu$  converging to  $w$  such that  $(z, w_\nu) \in E_\nu$  for every  $\nu \in \mathbb{N}$ .
- 3) For each  $z \in \mathbb{C}^{n-1}$  the set  $\mathcal{E}_z := \mathcal{E} \cap (\{z\} \times \mathbb{C})$  has zero 2-dimensional Lebesgue measure.

**Proof.** Let  $\Delta_R := \overline{B}^{n-1}(0, R) \times \mathbb{C}$ . For every  $(z, w_j^{(\nu+\mu)}(z)) \in E_{\nu+\mu} \cap \overline{\Delta}_R$  there exists  $(z, w_k^{(\nu)}(z)) \in E_\nu \cap \overline{\Delta}_R$  such that for suitably chosen signs one has

$$w_j^{(\nu+\mu)}(z) = w_k^{(\nu)}(z) + \sum_{l=\nu+1}^{\nu+\mu} \pm \varepsilon_l \sqrt{|z_{[l]} - a_l|}$$

(here, by some abuse of notation,  $\sqrt{\cdot}$  denotes a single-valued branch of the multi-valued function  $\sqrt{\cdot}$ ). By assumption we have  $\varepsilon_l \sqrt{|z_{[l]} - a_l|} = \varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^l$  on  $\overline{B}^{n-1}(0, R)$  for each  $l > \nu$ . Hence  $|w_j^{(\nu+\mu)}(z) - w_k^{(\nu)}(z)| < 1/2^\nu$  and it follows that the Hausdorff distance between  $E_{\nu+\mu} \cap \overline{\Delta}_R$  and  $E_\nu \cap \overline{\Delta}_R$  is less than  $1/2^\nu$ . In particular  $\{E_\nu \cap \overline{B}^n(0, R)\}_{\nu=1}^\infty$  is a Cauchy sequence in the Hausdorff metric and thus converges to a nonempty closed subset  $\mathcal{E}_{(R)} \subset \mathbb{C}^n$ . Since  $\mathcal{E} \cap \overline{B}^n(0, R) = \mathcal{E}_{(R)}$  for all  $R > 0$ , we conclude that  $\mathcal{E}$  is closed. Obviously it is also unbounded and

nonempty. The characterization of  $(z, w) \in \mathcal{E}$  as a limit of points  $(z, w_\nu) \in E_\nu$  follows immediately from the facts that in each bounded neighbourhood of  $(z, w)$  the set  $\mathcal{E}$  is the limit of  $\{E_\nu\}$  in the Hausdorff metric and that  $E_\nu \cap (\{z\} \times \mathbb{C}) \neq \emptyset$  for all  $z \in \mathbb{C}^{n-1}$ . Finally, by what we have already proven, we know that the Hausdorff distance between  $E_\nu \cap \overline{\Delta}_R$  and  $\mathcal{E}_{(R)}$  is not greater than  $1/2^\nu$ . Hence if  $z \in \mathbb{C}^{n-1}$  is fixed, the set  $\mathcal{E}_z$  is contained in  $\{z\} \times \bigcup_{j=1}^{2^\nu} \overline{\Delta}^1(w_j^{(\nu)}(z), 1/2^\nu)$  for every  $\nu \in \mathbb{N}$  big enough (here  $\overline{\Delta}^1(a, r) \subset \mathbb{C}$  denotes the closed disc centered at the point  $a$  of radius  $r$ ). But the volume of the later set is not greater than  $\pi/2^\nu$ , thus  $\mathcal{E}_z$  has zero 2-dimensional Lebesgue measure.  $\square$

If  $\{\varepsilon_l\}$  converges to zero fast enough, then by the previous lemma the analytic sets  $E_\nu$  determine a limit set  $\mathcal{E}$ . We want to use this set in the construction of our example. To do so we need to have two specific properties of this set. Namely, we want to ensure that  $\mathcal{E}$  has no analytic structure and we seek a description of  $\mathcal{E}$  in terms of certain sublevel sets of the polynomials  $P_\nu$ . In the next two sections we will show that we indeed can assure  $\mathcal{E}$  to have these properties, provided that  $\{\varepsilon_l\}$  is converging to zero fast enough.

### 3. Choice of the sequence $\{\varepsilon_l\}$ - Part I

First we want to show that for  $\{\varepsilon_l\}$  decreasing fast enough the set  $\mathcal{E}$  contains no analytic varieties of positive dimension. In order to do so it obviously suffices to show that  $\mathcal{E}$  contains no analytic disc, i.e. there exists no (nonconstant) holomorphic mapping  $f: \mathbb{D} \rightarrow \mathbb{C}^n$  from the unit disc  $\mathbb{D} \subset \mathbb{C}$  to  $\mathbb{C}^n$  with image completely contained in  $\mathcal{E}$ . For analytic discs with constant  $z$ -coordinates this is immediately clear, since we know that  $\mathcal{E}_z$  has zero two-dimensional Lebesgue measure for every  $z \in \mathbb{C}^{n-1}$ . The hard part is to show that there exists no analytic disc  $f(\mathbb{D}) \subset \mathcal{E}$  such that the projection  $f_z := \pi_z \circ f$  onto  $\mathbb{C}_z^{n-1}$  is not constant. The general idea is the following: Let  $f: \mathbb{D} \rightarrow \mathbb{C}^n$  be an analytic disc lying in the analytic hypersurface  $w = \sqrt{z_p - a}$ ,  $a \in \mathbb{C}$ , and such that  $f_z: \mathbb{D} \rightarrow \mathbb{C}_z^{n-1}$  is a biholomorphic embedding of  $\mathbb{D}$  into  $\mathbb{C}_z^{n-1}$ . Then  $f_z(\mathbb{D})$  is either completely contained in the slice  $S_a^p := \{z \in \mathbb{C}^{n-1} : z_p = a\}$  or does not intersect  $S_a^p$  at all. This is due to the fact that if  $S_a^p \cap f_z(U) = \{z_0\}$ ,  $U \subset \mathbb{D}$  open and small enough, then for the canonical parametrization  $g: f_z(U) \rightarrow \mathbb{C}_w$  of  $f(U)$  and for  $\zeta^+, \zeta^- \in \mathbb{C}^{n-1}$  such that  $z_0 + \zeta^+, z_0 - \zeta^- \in f_z(U)$  the slope  $|g(z_0 + \zeta^+) - g(z_0 - \zeta^-)| / \|\zeta^+ + \zeta^-\|$  becomes unbounded as  $\zeta^+, \zeta^- \rightarrow 0$  which contradicts the holomorphicity of  $g$ . Since each set  $E_\nu$  is defined by a sum of terms of the form  $\sqrt{z_{[l]} - a_l}$ , and since, moreover, the subsequence  $\{a_l^p\}_{l=1}^\infty$  of  $\{a_l\}$  is dense in  $\mathbb{C}$ , this will enable us to show that for  $\{\varepsilon_l\}$  decreasing fast enough every analytic disc  $f(\mathbb{D}) \subset \mathcal{E}$  must have constant  $z_p$ -coordinate. Due to the fact that  $p \in \mathbb{N}_{n-1}$  here is arbitrary, our assertion will be proved.

There arise some technical difficulties, the most important of which is the following: while for every described above analytic disc in the analytic hypersurface

$w = \sqrt{z_p - a}$  the projection  $f_z(\mathbb{D})$  cannot intersect  $S_a^p$  (at least if its  $z_p$ -coordinate is not already constant), this property might get spoiled when adding further terms  $\sqrt{z[l] - a_l}$ ,  $l \in \mathbb{N}$ , and thus does not carry over necessarily to the limit set  $\mathcal{E}$ . In general this problem can be easily handled, except, however, at points  $z_0 \in S_a^p$  that are contained in  $S_{a_l}^{[l]}$  for more than one  $l \in \mathbb{N}$ . In this situation there are root branches originating from  $z_0$  in different directions  $p_1, \dots, p_T \in \mathbb{N}_{n-1}$ , and in general their slopes near the point  $z_0$  may cancel out each other. To deal with this problem we will show that we can at least guarantee the following: for every  $z_0 \in S_{a_l}^{[l]} \cap B^{n-1}(0, l)$ ,  $l \in \mathbb{N}$ , there does not exist any analytic disc  $f(\mathbb{D}) \subset \mathcal{E}$  such that  $f_z(\mathbb{D}) \cap S_{a_l}^{[l]} = \{z_0\}$  and such that  $f_z(\mathbb{D})$  is contained in the cone  $z_0 + \bigcap_{t=1}^T \Gamma^{p_t}(\alpha)$ ; here

$$\Gamma^p(\alpha) := \{\zeta \in \mathbb{C}^{n-1} : \zeta_p \neq 0 \text{ and } \frac{|\zeta_q|}{|\zeta_p|} < \alpha \text{ for all } q \in \mathbb{N}_{n-1}, q \neq p\},$$

where  $\alpha$  is a positive number that will depend on the choice of  $\{\varepsilon_l\}$  (note that if  $\zeta \in \Gamma^p(\alpha)$  then also  $\lambda\zeta \in \Gamma^p(\alpha)$  for every  $\lambda \in \mathbb{C}^*$ ). In fact the faster  $\{\varepsilon_l\}$  decreases the bigger we will be able to choose  $\alpha$ . It turns out that this weaker assertion is sufficient for our purpose, since locally for every analytic disc  $f(\mathbb{D}) \subset \mathcal{E}$  the projection  $f_z(\mathbb{D})$  lies in  $\bigcap_{t=1}^T \Gamma^{p_t}(\alpha)$  for suitable  $p_1, \dots, p_T \in \mathbb{N}_{n-1}$  and  $\alpha > 0$  big enough.

The above complications, as well as most of the other technical difficulties for choosing the sequence  $\{\varepsilon_l\}$ , do not occur in the case  $n = 2$ . In fact in this case the proof becomes relatively simple and most of the work of this section is not needed. Hence in what follows we will often implicitly assume that  $n \geq 3$ , though this will not have any influence on the course and correctness of our arguments (for example the set  $\Gamma^p(\alpha) = \mathbb{C}^*$  is still well-defined for  $n = 2$ , though it is obviously not needed in this case).

**Remark.** Many of the statements in this section involve the function  $\sqrt{\cdot} : \mathbb{C} \rightarrow \mathbb{C}$  which is multivalued. In general, whenever such a statement is made, we will implicitly mean it to hold true for every choice of a single-valued branch  $(\sqrt{\cdot})_b : \mathbb{C} \rightarrow \mathbb{C}$  of  $\sqrt{\cdot}$  (no assumptions on continuity). However, there will be cases when we will have to deal with particular single-valued branches of  $\sqrt{\cdot}$ . By some abuse of notation they will be denoted by the same symbol  $\sqrt{\cdot}$ . We will always point out when  $\sqrt{\cdot}$  denotes a particular single-valued branch whenever such a situation first occurs.

**Lemma 3.** *There exists a constant  $0 < C < 1$  such that for all  $z, z', \zeta \in \mathbb{C}$*

$$\sqrt{|\zeta|} \leq \left| \sqrt{z + \zeta} - \sqrt{z' - \zeta} \right| \leq 2\sqrt{|\zeta|} \quad \text{if } |z|, |z'| \leq C|\zeta|.$$

**Proof.** This is immediately clear, since

$$\frac{\left| \sqrt{z + \zeta} - \sqrt{z' - \zeta} \right|}{\sqrt{|\zeta|}} = \left| \sqrt{(z/\zeta) + 1} - \sqrt{(z'/\zeta) - 1} \right| \xrightarrow{z/\zeta, z'/\zeta \rightarrow 0} \sqrt{2}. \quad \square$$

**Lemma 4.** For every  $p \in \mathbb{N}_{n-1}$  and  $\alpha > 0$  one has

$$\lim_{\zeta \rightarrow 0} \frac{|\sqrt{\zeta_p} - \sqrt{-\zeta_p}|}{2\|\zeta\|} = +\infty \quad \text{on } \Gamma^p(\alpha).$$

**Proof.** Indeed with  $c_\alpha := \max\{1, \alpha\}$  we have

$$\frac{|\sqrt{\zeta_p} - \sqrt{-\zeta_p}|}{2\|\zeta\|} = \frac{1}{\sqrt{2}} \frac{\sqrt{|\zeta_p|}}{\|\zeta\|} = \frac{1}{\sqrt{2}} \left( \sum_{q=1}^{n-1} \frac{|\zeta_q|^2}{|\zeta_p|} \right)^{-1/2} \geq \frac{1}{\sqrt{2}} \left( \sum_{q=1}^{n-1} c_\alpha |\zeta_q| \right)^{-1/2}$$

on  $\Gamma^p(\alpha)$  and the last term tends to  $+\infty$  as  $\zeta \rightarrow 0$ .  $\square$

**Lemma 5.** Let  $P := \{p_t\}_{t=1}^T \subset \{1, \dots, n-1\}$ ,  $p_t \neq p_{t'}$  if  $t \neq t'$ , and  $\{e_t\}_{t=1}^T \subset (0, \infty)$ ,  $T \geq 2$ . Define a constant  $\alpha > 0$  by  $\alpha := \min \left\{ \frac{1}{9} (e_m/e_{m+1})^2 : 1 \leq m \leq T-1 \right\}$ . Then for every  $\nu > 0$  there exists a positive number  $\delta > 0$  such that

$$\frac{\left| \sum_{m=1}^T e_m \left( \sqrt{z_{p_m} + (\zeta_{p_m} + \zeta'_{p_m})} - \sqrt{z_{p_m} - (\zeta_{p_m} + \zeta''_{p_m})} \right) \right|}{2\|\zeta\|} > \nu$$

for every  $\zeta \in \left( \bigcap_{m=1}^T \Gamma^{p_m}(\alpha) \right) \cap B^{n-1}(0, \delta)$  and  $\zeta', \zeta'', z \in \Delta^{n-1}(0, (C/2)\|\zeta\|_P)$ . Here  $\|\zeta\|_P \in (0, \infty]^{n-1}$  is defined by  $(\|\zeta\|_P)_p = |\zeta_p|$  if  $p \in P$ ,  $(\|\zeta\|_P)_p = \infty$  if  $p \in \mathbb{C}P := \mathbb{N}_{n-1} \setminus P$ , and  $\Delta^{n-1}(0, (r_1, \dots, r_{n-1})) := \{z \in \mathbb{C}^{n-1} : |z_p| < r_p, \text{ if } r_p > 0, \text{ or } z_p = 0, \text{ if } r_p = 0, p \in \mathbb{N}_{n-1}\}$  for  $r \in [0, \infty]^{n-1}$ .

**Remark.** The statement of this lemma is interesting and will be used only in the case when  $\alpha > 1$  (otherwise the intersection  $\bigcap_m \Gamma^{p_m}(\alpha)$  is empty).

**Proof.** For every  $m \in \mathbb{N}_{T-1}$  we define  $\alpha_m := \frac{1}{9} (e_m/e_{m+1})^2$  and for every  $m \in \mathbb{N}_T$  we let  $D_m(\zeta) := \{z \in \mathbb{C}^{n-1} : |z_{p_m}| \leq C\|\zeta_{p_m}\|\}$ . We will show by induction that for every  $t = 1, \dots, T$  the inequality

$$\left| \sum_{m=1}^t e_m \left( \sqrt{z'_{p_m} + \zeta_{p_m}} - \sqrt{z''_{p_m} - \zeta_{p_m}} \right) \right| \geq e_t \sqrt{|\zeta_{p_t}|} \quad (1)$$

holds true for  $\zeta \in \bigcap_{m=1}^{t-1} \Gamma^{p_m}(\alpha_m)$  and  $z', z'' \in \bigcap_{m=1}^t D_m(\zeta)$ . Indeed, the case  $t = 1$  is already proven by Lemma 3. For the step  $t \rightarrow t+1$  let  $H_{t+1}$  denote the left term in (1) where the sum is taken up to  $t+1$ . Using the induction hypothesis and applying Lemma 3 we see that

$$\begin{aligned} H_{t+1} &\geq \left| \sum_{m=1}^t e_m \left( \sqrt{z'_{p_m} + \zeta_{p_m}} - \sqrt{z''_{p_m} - \zeta_{p_m}} \right) \right| - e_{t+1} \left| \sqrt{z'_{p_{t+1}} + \zeta_{p_{t+1}}} - \sqrt{z''_{p_{t+1}} - \zeta_{p_{t+1}}} \right| \\ &\geq e_t \sqrt{|\zeta_{p_t}|} - 2e_{t+1} \sqrt{|\zeta_{p_{t+1}}|} \quad \text{for } \zeta \in \bigcap_{m=1}^{t-1} \Gamma^{p_m}(\alpha_m), \quad z', z'' \in \bigcap_{m=1}^{t+1} D_m(\zeta). \end{aligned}$$



Observe that there is nothing to show in the case  $\zeta_{p_{t+1}} = 0$ . Hence we can assume  $\zeta_{p_{t+1}} \neq 0$  and write

$$e_t \sqrt{|\zeta_{p_t}|} - 2e_{t+1} \sqrt{|\zeta_{p_{t+1}}|} = 2e_{t+1} \sqrt{|\zeta_{p_{t+1}}|} \left( \frac{e_t}{2e_{t+1}} \frac{\sqrt{|\zeta_{p_t}|}}{\sqrt{|\zeta_{p_{t+1}}|}} - 1 \right).$$

One immediately checks that the term between the brackets is not less than  $1/2$  precisely if  $|\zeta_{p_{t+1}}|/|\zeta_{p_t}| \leq \alpha_t$ , hence

$$e_t \sqrt{|\zeta_{p_t}|} - 2e_{t+1} \sqrt{|\zeta_{p_{t+1}}|} \geq e_{t+1} \sqrt{|\zeta_{p_{t+1}}|} \quad \text{for } \zeta \in \Gamma^{p_t}(\alpha_t).$$

This completes the induction and proves (1). But from Lemma 4 we know that

$$\lim_{\zeta \rightarrow 0} \frac{|\sqrt{\zeta_{p_T}} - \sqrt{-\zeta_{p_T}}|}{2\|\zeta\|} = +\infty \quad \text{on } \Gamma^{p_T}(\alpha_T),$$

where  $\alpha_T := \alpha$ . Combining this with the estimate (1) in the case  $t = T$  we conclude that for every  $\nu > 0$  there exists  $\delta > 0$  such that

$$\frac{\left| \sum_{m=1}^T e_m \left( \sqrt{z'_{p_m} + \zeta_{p_m}} - \sqrt{z''_{p_m} - \zeta_{p_m}} \right) \right|}{2\|\zeta\|} > \nu$$

for  $\zeta \in \bigcap_{m=1}^T \Gamma^{p_m}(\alpha_m) \cap B^{n-1}(0, \delta)$  and  $z', z'' \in \bigcap_{m=1}^T D_m(\zeta) = \Delta^{n-1}(0, C|\zeta|_P)$ . Since  $\alpha \leq \alpha_m$  for all  $m \in \mathbb{N}_T$  and  $\Gamma^p(\alpha) \subset \Gamma^p(\alpha')$  for  $\alpha \leq \alpha'$  this concludes the proof. Indeed, for  $\zeta', \zeta'', z \in \Delta^{n-1}(0, (C/2)|\zeta|_P)$  the points  $z' := z + \zeta'$  and  $z'' := z - \zeta''$  always satisfy  $z', z'' \in \Delta^{n-1}(0, C|\zeta|_P)$ .  $\square$

We want to estimate the slope between two points of the set  $E_\nu$  when their projection to  $\mathbb{C}_z^{n-1}$  lies near the zero set of one of the functions  $\sqrt{z_{[l]} - a_l}$ ,  $l = 1, \dots, \nu$ . For this we need some notations: for every  $\nu \in \mathbb{N}$  and  $p \in \mathbb{N}_{n-1}$  we define

$$S_\nu := \{\zeta \in \mathbb{C}^{n-1} : \zeta_{[\nu]} = a_\nu\}, \quad S^p := \{\zeta \in \mathbb{C}^{n-1} : \zeta_p = 0\}$$

and

$$L_\nu^p := \{l \in \mathbb{N} : 1 \leq l \leq \nu, [l] = p\}, \quad A_\nu^p := \{a_l \in \mathbb{C} : l \in L_\nu^p\}.$$

Obviously  $\bigcup_{p=1}^{n-1} L_\nu^p = \mathbb{N}_\nu$ . Moreover, if  $z \in \mathbb{C}^{n-1}$ , we define

$$L_\nu^p(z) := \{l \in L_\nu^p : z_p = a_l\}.$$

Note that  $L_\nu^p(z)$  consists of at most one element. Further for  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$  and  $z \in S_\nu$  we let

$$\mathcal{L}_\nu^P(z) := \bigcup_{p \in P} L_\nu^p(z).$$

Observe that under the assumptions on  $P$  and  $z$  we always have  $\nu \in \mathcal{L}_\nu^P(z)$ . As mentioned before the case  $|\mathcal{L}_\nu^P(z)| > 1$  is of special interest and leads us to consider the sets  $\bigcap_p \Gamma^p(\alpha)$  for  $\alpha > 1$ . Here  $\alpha$  was claimed to depend on  $\{\varepsilon_l\}$  and we now

clarify this dependence by the following definition: for every  $\nu \in \mathbb{N}$ ,  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$  and every  $z \in S_\nu$  let  $\alpha_\nu^P(z)$  be the positive number

$$\alpha_\nu^P(z) := \begin{cases} \nu + 1 & \text{if } \mathcal{L}_\nu^P(z) = \{\nu\} \\ \min \left\{ \frac{1}{9}(\varepsilon_l/\varepsilon_{l'})^2 : l, l' \in \mathcal{L}_\nu^P(z), l' > l \right\} & \text{if } \mathcal{L}_\nu^P(z) \supsetneq \{\nu\}. \end{cases}$$

Observe that, since the sequence  $\{\varepsilon_l\}$  is still in our hands, we can always assume that  $\alpha_\nu^P(z) > 1$ . Finally for each  $P \subset \mathbb{N}_{n-1}$  and  $\alpha > 0$  we let

$$\gamma(P, \alpha) := \left( \bigcap_{p \in P} \Gamma^p(\alpha) \right) \cap \left( \bigcap_{p \in \mathbb{C}P} S^p \right).$$

**Lemma 6.** *Suppose  $\varepsilon_1, \dots, \varepsilon_\nu$  have already been chosen. Let  $\delta > 0$ . Then for every  $z_0 \in S_\nu$  and  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$  there exist  $r^P(z_0) > 0$  and  $\delta^P(z_0) \in (0, \delta)$  such that for every  $j, k \in \mathbb{N}_{2\nu}$  the inequality*

$$\frac{|w_j^{(\nu)}(z + (\zeta + \zeta')) - w_k^{(\nu)}(z - (\zeta + \zeta''))|}{2\|\zeta\|} > \nu \quad (2)$$

holds for every  $z \in B^{n-1}(z_0, r^P(z_0))$ ,  $\zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap \partial B^{n-1}(0, \delta^P(z_0))$  and  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ; here  $|\zeta| = (|\zeta_1|, \dots, |\zeta_{n-1}|)$ .

**Proof.** Fix  $z_0 \in S_\nu$  and  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$ . For each  $p \in \mathbb{N}_{n-1}$  let  $U_p \subset \mathbb{C}$  be an open convex neighbourhood of  $z_{0,p}$  such that

$$U_p \cap A_\nu^p = \begin{cases} \emptyset & \text{if } L_\nu^p(z_0) = \emptyset \\ z_{0,p} & \text{if } L_\nu^p(z_0) \neq \emptyset \end{cases}$$

and let  $U := U_1 \times \dots \times U_{n-1}$ . Choose  $r > 0$  so small that  $B^{n-1}(z_0, 2r) \subset U$ . For each  $l \in \mathbb{N}_\nu$  consider a single-valued branch of the multi-valued function  $\sqrt{z_{[l]} - a_l}$  which will also be denoted here by  $\sqrt{z_{[l]} - a_l}$ . Since for every  $l \in \mathbb{N}_\nu \setminus \bigcup_{p=1}^{n-1} L_\nu^p(z_0)$  we have  $a_l \notin U_{[l]}$ , we can assume that  $\sqrt{z_{[l]} - a_l}$  is holomorphic on  $U$  for these  $l$ . After possibly changing the numeration of the roots of  $P_\nu(z, \cdot)$  for  $z \in U$  we may further assume for every  $h \in \mathbb{N}_{2\nu}$  that  $w_h^{(\nu)}(z) = \sum_{l=1}^\nu \pm \varepsilon_l \sqrt{z_{[l]} - a_l}$  for suitably chosen signs depending only on  $l$  and  $h$ . Now define  $\tilde{w}_h : B^{n-1}(z_0, 2r) \rightarrow \mathbb{C}$  as

$$\tilde{w}_h(z) := \sum_{p \in P} \sum_{l \in L_\nu^p \setminus L_\nu^p(z_0)} \pm \varepsilon_l \sqrt{z_{[l]} - a_l} + \sum_{p \in \mathbb{C}P} \sum_{l \in L_\nu^p} \pm \varepsilon_l \sqrt{z_{[l]} - a_l}. \quad (3)$$

Since  $\mathbb{N}_\nu = \bigcup_{p=1}^{n-1} L_\nu^p$ , we obviously have  $w_h^{(\nu)}(z) = \tilde{w}_h(z) + \sum_{l \in \mathcal{L}_\nu^P(z_0)} \pm \varepsilon_l \sqrt{z_{[l]} - a_l}$  on  $B^{n-1}(z_0, 2r)$ . Let  $\mathbb{N}_{2\nu}^2 := \mathbb{N}_{2\nu} \times \mathbb{N}_{2\nu}$  and  $\mathbb{N}_{2\nu}^2(z_0) := \{(j, k) \in \mathbb{N}_{2\nu}^2 : \tilde{w}_j(z_0) = \tilde{w}_k(z_0)\}$ .

**STEP 1:** We show that there exist  $r' > 0$  and  $\delta' \in (0, \delta)$  such that (2) holds for every  $\zeta \in B^{n-1}(0, \delta')$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in B^{n-1}(z_0, r')$  and  $(j, k) \in \mathbb{N}_{2\nu}^2 \setminus \mathbb{N}_{2\nu}^2(z_0)$ .

For  $l \in L_\nu^p(z_0)$  we have  $z_{0,[l]} = a_l$  and  $\sqrt{\cdot}$  is continuous at the origin, hence we conclude from (3) and the holomorphicity of  $\sqrt{z_{[l]} - a_l}$  for  $l \in L_\nu^p \setminus L_\nu^p(z_0)$  that  $\tilde{w}_h$

is continuous at  $z_0$  for every  $h \in \mathbb{N}_{2\nu}$ . Thus there exist  $M > 0$  and  $r_1 > 0$  such that  $|\tilde{w}_j(z + (\zeta + \zeta')) - \tilde{w}_k(z - (\zeta + \zeta''))| > M$  for every  $\zeta \in B^{n-1}(0, r_1)$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in B^{n-1}(z_0, r_1)$  and  $(j, k) \in \mathbb{N}_{2\nu}^2 \setminus \mathbb{N}_{2\nu}^2(z_0)$ . Moreover, since again  $z_{0,[l]} = a_l$  for  $l \in \mathcal{L}_\nu^P(z_0)$  and  $\sqrt{\cdot}$  is continuous at the origin, there exists  $r_2 > 0$  such that  $\sqrt{|(z_{[l]} \pm (\zeta_{[l]} + \tilde{\zeta})) - a_l|} < M/(4(n-1)\varepsilon_l)$ , where  $\tilde{\zeta} \in \{\zeta', \zeta''\}$ , for every  $\zeta \in B^{n-1}(0, r_2)$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in B^{n-1}(z_0, r_2)$  and  $l \in \mathcal{L}_\nu^P(z_0)$ . Let  $r' := \min\{r, r_1, r_2\}$  and  $\delta' := \min\{\delta, r, r_1, r_2, M/4\nu\}$ . Then the following estimate holds true for every  $\zeta \in B^{n-1}(0, \delta')$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in B^{n-1}(z_0, r')$  and  $(j, k) \in \mathbb{N}_{2\nu}^2 \setminus \mathbb{N}_{2\nu}^2(z_0)$ :

$$\begin{aligned} & \frac{|w_j^{(\nu)}(z + (\zeta + \zeta')) - w_k^{(\nu)}(z - (\zeta + \zeta''))|}{2\|\zeta\|} \geq \frac{|\tilde{w}_j(z + (\zeta + \zeta')) - \tilde{w}_k(z - (\zeta + \zeta''))|}{2\|\zeta\|} - \\ & - \frac{\sum_{l \in \mathcal{L}_\nu^P(z_0)} \varepsilon_l \left( \sqrt{|(z_{[l]} + (\zeta_{[l]} + \zeta'_{[l]})) - a_l|} + \sqrt{|(z_{[l]} - (\zeta_{[l]} + \zeta''_{[l]})) - a_l|} \right)}{2\|\zeta\|} \\ & > \frac{M - \sum_{l \in \mathcal{L}_\nu^P(z_0)} 2\varepsilon_l M/(4(n-1)\varepsilon_l)}{2\|\zeta\|} \geq \frac{M - M/2}{2\|\zeta\|} > \nu. \end{aligned}$$

STEP 2: We show that there exist  $r'' \in (0, r')$  and  $\delta'' \in (0, \delta')$  such that (2) holds for every  $\zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap K^{n-1}(\delta''/2, \delta'')$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in \Delta^{n-1}(z_0, (C/2)|\zeta|_P) \cap B^{n-1}(z_0, r'')$  and  $(j, k) \in \mathbb{N}_{2\nu}^2$ , where for  $R_1, R_2 \geq 0$  we put  $K^{n-1}(R_1, R_2) := \{z \in \mathbb{C}^{n-1} : R_1 < \|z\| < R_2\}$ .

Observe that the first term in (3) is holomorphic in  $B^{n-1}(z_0, 2r)$  and the second term is constant on the set  $z_0 + \bigcap_{p \in \mathbb{C}P} S^p$ . Therefore we can find  $M > 0$  and  $\tilde{r} > 0$  such that

$$\frac{|\tilde{w}_j(z_0 + (\zeta + \zeta')) - \tilde{w}_k(z_0 - (\zeta + \zeta''))|}{2\|\zeta\|} < M$$

for all  $\zeta \in (\bigcap_{p \in \mathbb{C}P} S^p) \cap B^{n-1}(0, \tilde{r})$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$  and  $(j, k) \in \mathbb{N}_{2\nu}^2(z_0)$ . Moreover, since  $z_{0,[l]} = a_l$  for every  $l \in \mathcal{L}_\nu^P(z_0)$ , we have  $\sqrt{|(z_{[l]} \pm (\zeta_{[l]} + \tilde{\zeta}_{[l]})) - a_l|} = \sqrt{|(z_{[l]} - z_{0,[l]}) \pm (\zeta_{[l]} + \tilde{\zeta}_{[l]})|}$ , where  $\tilde{\zeta} \in \{\zeta', \zeta''\}$ . Hence, using Lemma 3 and 4 if  $\mathcal{L}_\nu^P(z_0) = \{\nu\}$  and Lemma 5 if  $\mathcal{L}_\nu^P(z_0) \supsetneq \{\nu\}$ , there exists  $\tilde{\delta} > 0$  such that

$$\frac{\left| \sum_{l \in \mathcal{L}_\nu^P(z_0)} \varepsilon_l \left( \sqrt{|(z_{[l]} + (\zeta_{[l]} + \zeta'_{[l]})) - a_l|} - \sqrt{|(z_{[l]} - (\zeta_{[l]} + \zeta''_{[l]})) - a_l|} \right) \right|}{2\|\zeta\|} > \nu + M$$

for all  $\zeta \in [\bigcap_{p \in P} \Gamma^p(\alpha_\nu^P(z_0))] \cap B^{n-1}(0, \tilde{\delta})$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|_P)$  and  $z \in \Delta^{n-1}(z_0, (C/2)|\zeta|_P)$  (recall the definition of  $\alpha_\nu^P(z_0)$ ). Now choose  $\delta''$  such that  $0 < \delta'' < \min\{\tilde{r}, \tilde{\delta}, \delta'\}$ . Observe that  $\tilde{w}_h$  is continuous in  $z_0 + [\bigcap_{p \in \mathbb{C}P} S^p] \cap B^{n-1}(0, 2r)$  for every  $h \in \mathbb{N}_{2\nu}$ . Hence there exists  $r'' \in (0, r')$  such that the following estimate holds true for every  $\zeta \in (\bigcap_{p \in \mathbb{C}P} S^p) \cap K^{n-1}(\delta''/2, \delta'')$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,

$z \in B^{n-1}(z_0, r'')$  and  $(j, k) \in \mathbb{N}_{2\nu}^2(z_0)$ :

$$\frac{|\tilde{w}_j(z + (\zeta + \zeta')) - \tilde{w}_k(z - (\zeta + \zeta''))|}{2\|\zeta\|} < M.$$

Thus for every  $\zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap K^{n-1}(\delta''/2, \delta'')$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in \Delta^{n-1}(z_0, (C/2)|\zeta|_P) \cap B^{n-1}(z_0, r'')$  and  $(j, k) \in \mathbb{N}_{2\nu}^2$  we get

$$\begin{aligned} & \frac{|w_j^{(\nu)}(z + (\zeta + \zeta')) - w_k^{(\nu)}(z - (\zeta + \zeta''))|}{2\|\zeta\|} \\ & \geq \frac{\left| \sum_{l \in \mathcal{L}_\nu^P(z_0)} \varepsilon_l \left( \sqrt{(z_{[l]} + (\zeta_{[l]} + \zeta'_{[l]}))} - a_l - \sqrt{(z_{[l]} - (\zeta_{[l]} + \zeta''_{[l]}))} - a_l \right) \right|}{2\|\zeta\|} \\ & - \frac{|\tilde{w}_j(z + (\zeta + \zeta')) - \tilde{w}_k(z - (\zeta + \zeta''))|}{2\|\zeta\|} > \nu. \end{aligned}$$

STEP 3: We show that there exist  $r^P(z_0) > 0$  and  $\delta^P(z_0) \in (0, \delta)$  such that (2) holds for every  $\zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap \partial B^{n-1}(0, \delta^P(z_0))$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in B^{n-1}(z_0, r^P(z_0))$  and  $j, k \in \mathbb{N}_{2\nu}$ .

We already know that (2) holds for every  $\zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap K^{n-1}(\delta''/2, \delta'')$ ,  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ ,  $z \in \Delta^{n-1}(z_0, (C/2)|\zeta|_P) \cap B^{n-1}(z_0, r'')$  and  $j, k \in \mathbb{N}_{2\nu}$ . It only remains to make proper choices for the constants  $r^P(z_0)$  and  $\delta^P(z_0)$ . First choose any  $\delta^P(z_0)$  such that  $\delta'' > \delta^P(z_0) > \delta''/2$ . Then there exists  $K > 0$  such that

$$|\zeta_p| > K \quad \text{for all } \zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap \partial B^{n-1}(0, \delta^P(z_0)), p \in P.$$

Indeed, let  $p \in P$ . Then for  $\zeta \in \gamma(P, \alpha_\nu^P(z_0))$  we have in particular  $\zeta \in \Gamma^p(\alpha_\nu^P(z_0))$  and hence  $|\zeta_q|/|\zeta_p| < \alpha_\nu^P(z_0)$  for every  $q \in \mathbb{N}_{n-1}$  (assuming that  $\alpha_\nu^P(z_0) > 1$  which is the only interesting case). Thus  $\|\zeta\| < \alpha_\nu^P(z_0)\sqrt{n-1}|\zeta_p|$ . Since also  $\zeta \in \partial B^{n-1}(0, \delta^P(z_0))$ , we conclude that  $|\zeta_p| > \delta^P(z_0)/(\alpha_\nu^P(z_0)\sqrt{n-1}) =: K$ . Now choose  $\rho > 0$  such that  $|z_p - z_{0,p}| < (CK)/2$  for all  $z \in B^{n-1}(z_0, \rho)$  and  $p \in P$ , i.e.  $B^{n-1}(z_0, \rho) \subset \Delta^{n-1}(z_0, (C/2)|\zeta|_P)$  for all  $\zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap \partial B^{n-1}(0, \delta^P(z_0))$ . Then  $r^P(z_0) := \min\{r'', \rho\}$  is a constant as desired.  $\square$

Fix  $\nu \in \mathbb{N}$ . By the previous lemma we have assigned positive numbers  $r^P(z_0)$ ,  $\delta^P(z_0)$  to every  $z_0 \in S_\nu$ . As we shall see in the proof of Lemma 8 the choice of  $\varepsilon_{\nu+1}$  will depend on the numbers  $\delta^P(z_0)$ ,  $z_0 \in S_\nu$ ; in fact we will need a positive lower bound for the set  $\{\delta^P(z_0) : z_0 \in S_\nu\}$ . However, such a bound does not always exist. Hence from now on we restrict our attention to the compact subset  $S_\nu \cap \overline{B}^{n-1}(0, \nu)$  of  $S_\nu$ . This set can be covered by finitely many balls  $B^{n-1}(z_0, r^P(z_0))$ ,  $z_0 \in S_\nu$ , and thus leads to a finite set  $\{\delta^P(z_1), \dots, \delta^P(z_m)\} \subset (0, \infty)$  (which of course has a positive minimum). On the way we have to choose the numbers  $r^P(z_0)$  in the

covering  $\{B^{n-1}(z_0, r^P(z_0))\}_{z_0 \in S_\nu}$  carefully in order to limit the influence of points  $z_0 \in S_\nu$  with small value  $\alpha_\nu^P(z_0)$ . For this purpose we need some further notations: fix a decreasing sequence  $\{\rho_\nu\}$  of positive numbers converging to zero, such that

$$\max_{1 \leq p \leq n-1} \text{vol} \left( \bigcup_{l \in L_\nu^p} \Delta^1(a_l, \rho_\nu) \right) \rightarrow 0 \quad \text{for } \nu \rightarrow \infty.$$

Then for every  $\nu \in \mathbb{N}$ ,  $p \in \mathbb{N}_{n-1}$  and  $z \in \mathbb{C}^{n-1}$  we let

$$\tilde{L}_\nu^p(z) := \{l \in L_\nu^p : |z_p - a_l| \leq \rho_\nu\}.$$

Moreover if  $z \in S_\nu$  and  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$  we let

$$\tilde{\mathcal{L}}_\nu^P(z) := \bigcup_{p \in P} \tilde{L}_\nu^p(z).$$

Note that under the assumptions on  $P$  and  $z$  we always have  $\nu \in \tilde{\mathcal{L}}_\nu^P(z)$ . Hence

$$\tilde{\alpha}_\nu^P(z) := \begin{cases} \nu + 1 & \text{if } \tilde{\mathcal{L}}_\nu^P(z) = \{\nu\} \\ \min \left\{ \nu + 1, \min \left\{ \frac{1}{9}(\varepsilon_l/\varepsilon_{l'})^2 : l, l' \in \tilde{\mathcal{L}}_\nu^P(z), l' > l \right\} \right\} & \text{if } \tilde{\mathcal{L}}_\nu^P(z) \supsetneq \{\nu\} \end{cases}$$

is a well-defined positive number.

**Corollary 7.** *Suppose  $\varepsilon_1, \dots, \varepsilon_\nu$  have already been chosen. Let  $\delta > 0$ . Then there exists a finite subset  $D_\nu := \{\delta_\nu^1, \dots, \delta_\nu^{d_\nu}\} \subset (0, \delta)$  such that for every  $z \in S_\nu \cap B^{n-1}(0, \nu)$  and  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$  there exists some  $\sigma \in \{1, \dots, d_\nu\}$  such that for every  $j, k \in \mathbb{N}_{2^\nu}$  the inequality*

$$\frac{|w_j^{(\nu)}(z + (\zeta + \zeta')) - w_k^{(\nu)}(z - (\zeta + \zeta''))|}{2\|\zeta\|} > \nu \quad (4)$$

holds true for all  $\zeta \in \gamma(P, \tilde{\alpha}_\nu^P(z)) \cap \partial B^{n-1}(0, \delta_\nu^\sigma)$  and  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)\|\zeta\|)$ .

**Proof.** By the previous lemma for every  $z_0 \in S_\nu$  and  $P \subset \mathbb{N}_{n-1}$ ,  $[\nu] \in P$ , there exist positive numbers  $r^P(z_0) \in (0, \rho_\nu)$  and  $\delta^P(z_0) \in (0, \delta)$  such that (4) holds for every  $j, k \in \mathbb{N}_{2^\nu}$ ,  $z \in B^{n-1}(z_0, r^P(z_0))$ ,  $\zeta \in \gamma(P, \alpha_\nu^P(z_0)) \cap \partial B^{n-1}(0, \delta^P(z_0))$  and  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)\|\zeta\|)$ . Let

$$r(z_0) := \min \{r^P(z_0) : P \subset \mathbb{N}_{n-1} \text{ such that } [\nu] \in P\}.$$

By compactness of  $S_\nu \cap \overline{B}^{n-1}(0, \nu)$  there exist finitely many points  $z_1, \dots, z_M \in S_\nu$  such that  $S_\nu \cap \overline{B}^{n-1}(0, \nu) \subset \bigcup_{m=1}^M B^{n-1}(z_m, r(z_m))$ . Let

$$D_\nu := \{\delta^P(z_m) : P \subset \mathbb{N}_{n-1} \text{ such that } [\nu] \in P, m = 1, \dots, M\}.$$

Then for every  $z \in S_\nu \cap B^{n-1}(0, \nu)$  and  $P \subset \mathbb{N}_{n-1}$ ,  $[\nu] \in P$ , there exist  $\sigma \in \{1, \dots, d_\nu\}$  and  $m \in \mathbb{N}_M$  such that  $|z - z_m| \leq \rho_\nu$  and such that (4) holds for every  $j, k \in \mathbb{N}_{2^\nu}$ ,  $\zeta \in \gamma(P, \alpha_\nu^P(z_m)) \cap \partial B^{n-1}(0, \delta_\nu^\sigma)$  and  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)\|\zeta\|)$ . It remains to observe that we herein can replace  $\alpha_\nu^P(z_m)$  by  $\tilde{\alpha}_\nu^P(z)$ . Indeed, since  $|z - z_m| \leq \rho_\nu$ , we have  $L_\nu^p(z_m) \subset \tilde{L}_\nu^p(z)$  for all  $p \in \mathbb{N}_{n-1}$  and thus  $\mathcal{L}_\nu^P(z_m) \subset \tilde{\mathcal{L}}_\nu^P(z)$ . Recalling the definitions of  $\alpha_\nu^P(z_m)$  and  $\tilde{\alpha}_\nu^P(z)$  we conclude that  $\tilde{\alpha}_\nu^P(z) \leq \alpha_\nu^P(z_m)$ . In particular we get  $\gamma(P, \tilde{\alpha}_\nu^P(z)) \subset \gamma(P, \alpha_\nu^P(z_m))$ .  $\square$

We are now able to specify the choice of the sequence  $\{\varepsilon_l\}$ :

**Lemma 8.** *If  $\{\varepsilon_l\}$  is decreasing fast enough, then for every fixed  $\nu \in \mathbb{N}$  and for every  $z \in S_\nu \cap B^{n-1}(0, \nu)$  and  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$  there exists  $\delta \in (0, 1/\nu)$  such that*

$$\frac{w' - w''}{\|\zeta' + 2\zeta + \zeta''\|} \geq \frac{\nu - 1}{1 + (C/2)} \quad \text{for all } w' \in \mathcal{E}_{z+(\zeta+\zeta')}, w'' \in \mathcal{E}_{z-(\zeta+\zeta'')} \quad (5)$$

and all choices of  $\zeta \in \gamma(P, \tilde{\alpha}_\nu^P(z)) \cap \partial B^{n-1}(0, \delta)$  and  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$ . Moreover  $\frac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > l$  and  $\varepsilon_l \sqrt{|z_{[l]} - a_l|} < \frac{1}{2^l}$  on  $B^{n-1}(0, l)$ .

**Proof.** We proceed by induction on  $l$  and simultaneously choose a sequence  $(D_l)$  of finite subsets  $D_l = \{\delta_l^1, \dots, \delta_l^{d_l}\} \subset (0, 1/l)$  such that  $\varepsilon_l \sqrt{|z_{[l]} - a_l|} < \frac{1}{2^l} \min\{\delta' \in D_\nu : 1 \leq \nu \leq l-1\}$  for every  $z \in B^{n-1}(0, l+1)$ . First let  $\varepsilon_1 := 1$  and let  $D_1 \subset (0, 1)$  be the set provided by Corollary 7 in the case  $\nu, \delta = 1$ . If  $\varepsilon_1, \dots, \varepsilon_l$  and  $D_1, \dots, D_l$  have already been chosen, we choose  $\varepsilon_{l+1} > 0$  so small that  $\frac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > l$  and  $\varepsilon_{l+1} \sqrt{|z_{[l+1]} - a_{l+1}|} < \frac{1}{2^{l+1}} \min\{\delta' \in D_\nu : 1 \leq \nu \leq l\}$  for  $z \in B^{n-1}(0, l+2)$ . Observe that every  $\varepsilon'_{l+1} \in (0, \varepsilon_{l+1})$  would also be a proper choice for  $\varepsilon_{l+1}$ . We then take for  $D_{l+1}$  the set provided by Corollary 7 in the case  $\nu = l+1$  and  $\delta = 1/(l+1)$ .

Fix  $\nu \in \mathbb{N}$ ,  $z \in S_\nu \cap B^{n-1}(0, \nu)$  and  $P \subset \mathbb{N}_{n-1}$  such that  $[\nu] \in P$ . Then by choice of  $D_\nu$  there exists  $\delta \in D_\nu$  such that estimate (4) holds true for all  $j, k \in \mathbb{N}_{2^\nu}$  and all considered  $\zeta, \zeta', \zeta''$ . By choice of the sequence  $(\varepsilon_l)$ , if for abbreviation we write  $z^+ := z + (\zeta + \zeta')$  and  $z^- := z - (\zeta + \zeta'')$ , we thus get the following estimate for all  $\mu > \nu$  and  $j', k' \in \mathbb{N}_{2^\mu}$  (for suitable  $j, k \in \mathbb{N}_{2^\nu}$  depending on  $j', k'$ ):

$$\begin{aligned} & \left| \frac{w_{j'}^{(\mu)}(z + (\zeta + \zeta')) - w_{k'}^{(\mu)}(z - (\zeta + \zeta''))}{\|\zeta' + 2\zeta + \zeta''\|} \right| \geq \frac{|w_{j'}^{(\mu)}(z^+) - w_{k'}^{(\mu)}(z^-)|}{(2+C)\|\zeta\|} \\ & \geq \frac{|w_j^{(\nu)}(z^+) - w_k^{(\nu)}(z^-)|}{(2+C)\|\zeta\|} - \frac{1}{(2+C)\|\zeta\|} \sum_{l=\nu+1}^{\mu} \varepsilon_l \left( \sqrt{|z_{[l]}^+ - a_l|} + \sqrt{|z_{[l]}^- - a_l|} \right) \\ & \geq \frac{\nu}{1+(C/2)} - \frac{1}{(2+C)\delta} \sum_{l=\nu+1}^{\mu} \frac{\delta}{2^{l-1}} \geq \frac{\nu-1}{1+(C/2)}. \end{aligned}$$

Since by Lemma 2 each  $(z, w) \in \mathcal{E}$  is a limit of points  $(z, w_{j_\mu}^{(\mu)})$  this proves (5).  $\square$

**Lemma 9.** *If  $\{\varepsilon_l\}$  is decreasing fast enough, then  $\mathcal{E}$  contains no analytic variety of positive dimension.*

**Proof.** Let  $\{\varepsilon_l\}$  be decreasing so fast that the assertions of Lemma 8 hold true. To get a contradiction assume that  $\mathcal{E}$  contains an analytic variety of positive dimension. Then in particular  $\mathcal{E}$  contains a nonconstant analytic disc, i.e. there exists a nonconstant holomorphic mapping  $f = (f_1, f_2, \dots, f_n): \mathbb{D}_r(0) \rightarrow \mathbb{C}^n$  such that  $f(\mathbb{D}_r(0)) \subset \mathcal{E}$ , where  $\mathbb{D}_r(\xi_0) = \{\xi \in \mathbb{C} : |\xi - \xi_0| < r\}$ . Let  $P \subset \mathbb{N}_{n-1}$  be the set of all coordinate directions in  $\mathbb{C}^{n-1}$  such that  $f_p$  is not constant. Since by the

choice of  $\{\varepsilon_l\}$  and Lemma 2 the set  $\mathcal{E}_z$  has zero 2-dimensional Lebesgue measure for every  $z \in \mathbb{C}^{n-1}$ , we see that  $P \neq \emptyset$ . Without loss of generality we can assume that  $P = \{1, \dots, T\}$  for some  $T \leq n-1$ . After possibly passing to a subset  $\mathbb{D}_{r'}(\xi_0) \subset \mathbb{D}_r(0)$  we can assume by the implicit function theorem that there exist an open subset  $U \subset \mathbb{C}$  and some

$$\phi: U \rightarrow \mathbb{C}^n \text{ holomorphic, } \phi(U) = f(\mathbb{D}_{r'}(\xi_0))$$

such that  $\phi(\xi) = (\xi, \phi_2(\xi), \dots, \phi_T(\xi), q_{T+1}, \dots, q_{n-1}, \phi_n(\xi)) =: (\phi_*(\xi), \phi_n(\xi))$  with suitable constants  $q_{T+1}, \dots, q_{n-1} \in \mathbb{C}$ . After a possible shrinking of  $U$  we can assume that there exist positive numbers  $\sigma, \theta > 0$  such that on  $U$

$$\theta < |\phi'_p| \quad \text{for all } p \in \mathbb{N}_T, \quad |\phi'_p| < \sigma \quad \text{for all } p \in \mathbb{N}_n. \quad (6)$$

Indeed  $\theta$  exists since the zero set of each  $|\phi'_p|$  is discret and we use Cauchy's estimates to find  $\sigma$ . Thus after possibly shrinking  $U$  again, we can assume that for  $z, z' \in \phi(U)$  and  $1 \leq s, t \leq T$  we have  $\theta < |z'_t - z_t|/|z'_1 - z_1|$  and  $|z'_s - z_s|/|z'_1 - z_1| < \sigma$ , i.e.  $|z'_s - z_s|/|z'_t - z_t| < \sigma/\theta$ . In particular we see that there exists  $\alpha := \sigma/\theta > 1$  such that

$$D\phi_*(z_1)(\mathbb{C}) \subset \gamma(P, \alpha) \quad \text{for all } z_1 \in U.$$

Moreover (after possibly further shrinking  $U$ ) we can assume that for every  $p \in \mathbb{N}_n$

$$\frac{|\phi_p(a + \xi) - \phi_p(a) - \phi'_p(a)\xi|}{|\xi|} < (C/2)\theta \quad \text{for all } a \in U, \xi \in \mathbb{C} \text{ s.t. } a + \xi \in U. \quad (7)$$

Since we can assume  $f(\mathbb{D}_r(0))$  to be bounded, and since  $\frac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > l$ , we can choose  $\nu_0 \in \mathbb{N}$  so big that  $\phi_*(U) \subset B^{n-1}(0, \nu_0)$ ,

$$\nu_0 + 1 > \alpha \quad \text{and} \quad \frac{1}{9}(\varepsilon_l/\varepsilon_{l+1})^2 > \alpha \quad \text{for all } l \geq \nu_0. \quad (8)$$

Further, since  $\max_{1 \leq p \leq n-1} \text{vol}(\bigcup_{l \in L_\nu^p} \Delta^1(a_l, \rho_\nu)) \rightarrow 0$  for  $\nu \rightarrow \infty$  and  $\{\rho_\nu\}$  is decreasing, we can assume (after possibly enlarging  $\nu_0$  and then shrinking of  $U$ ) that  $\phi_p(U) \cap \bigcup_{l \in L_{\nu_0}^p} \Delta^1(a_l, \rho_\nu) = \emptyset$  for all  $p \in \mathbb{N}_T$  and  $\nu \geq \nu_0$ . But then  $\tilde{\mathcal{Z}}_\nu^P(z_p) \cap \mathbb{N}_{\nu_0} = \emptyset$  for all  $z_p \in \phi_p(U)$ ,  $p \in \mathbb{N}_T$ ,  $\nu \geq \nu_0$ . By definition of  $\tilde{\alpha}_\nu^P(z)$  and from (8) we therefore get

$$\tilde{\alpha}_\nu^P(z) > \alpha \quad \text{for all } \nu \geq \nu_0, z \in S_\nu \cap \phi_*(U), [\nu] \in P.$$

After these preparations we now choose a strictly increasing sequence  $\{\nu_k\}$  of natural numbers such that for each  $\nu$  from this sequence we have

$$\nu \geq \nu_0, \quad [\nu] = 1, \quad B^{n-1}(\phi_*(a_\nu), 1/\nu) \subset U \times \mathbb{C}^{n-2}.$$

Let  $\nu$  be an arbitrary fixed member of this sequence. Since  $\phi_*(U) \subset B^{n-1}(0, \nu_0)$  we see that  $z := \phi_*(a_\nu) \in S_\nu \cap B^{n-1}(0, \nu)$ . Hence we can use Lemma 8 to find a  $\delta \in (0, 1/\nu)$  such that

$$\frac{w' - w''}{\|\zeta' + 2\zeta + \zeta''\|} \geq \frac{\nu - 1}{1 + (C/2)} \quad \text{for all } w' \in \mathcal{E}_{z+(\zeta+\zeta')}, w'' \in \mathcal{E}_{z-(\zeta+\zeta'')}$$

and all choices of

$$\zeta \in \gamma(P, \tilde{\alpha}_\nu^P(z)) \cap \partial B^{n-1}(0, \delta) \quad \text{and} \quad \zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|). \quad (9)$$

By the choice of  $U$  respectively  $\nu_0$  we have  $D\phi_*(a_\nu)(\xi - a_\nu) \in \gamma(P, \alpha)$  for all  $\xi \in U \setminus \{a_\nu\}$  and  $\tilde{\alpha}_\nu^P(z) > \alpha$ , hence  $D\phi_*(a_\nu)(\xi - a_\nu) \in \gamma(P, \tilde{\alpha}_\nu^P(z))$ . Moreover  $B^{n-1}(z, \delta) \subset U \times \mathbb{C}^{n-2}$ . Thus

$$\Sigma := [z + \gamma(P, \tilde{\alpha}_\nu^P(z))] \cap [z + \{D\phi_*(a_\nu)(\xi - a_\nu) : \xi \in U, \xi \neq a_\nu\}] \cap \partial B^{n-1}(z, \delta)$$

is nonempty. Therefore we can choose  $\zeta \in \gamma(P, \tilde{\alpha}_\nu^P(z))$  such that  $z \pm \zeta \in \Sigma$ , and  $\xi \in \mathbb{C}$  such that  $a_\nu \pm \xi \in U$  and  $D\phi_*(a_\nu)(\pm \xi) = \pm \zeta$ . Now applying (7) in the case  $a = a_\nu$  and using (6) yields

$$|\phi_p(a_\nu + \xi) - z_p - \zeta_p| < (C/2)\theta|\xi| < (C/2)|\phi'_p(a_\nu)\xi| = (C/2)|\zeta_p|$$

for every  $p \in \mathbb{N}_T$ . Since also  $\phi_p(a_\nu + \xi) = z_p + \zeta_p$  for  $p \in \mathbb{N}_{n-1} \setminus \mathbb{N}_T$  and  $\phi_1(z_1) = z_1$ , this shows that there exist uniquely determined  $\zeta', \zeta'' \in \Delta^{n-1}(0, (C/2)|\zeta|)$  such that  $z + (\zeta + \zeta') = \phi_*(a_\nu + \xi)$ ,  $z - (\zeta + \zeta'') = \phi_*(a_\nu - \xi)$  and  $\zeta'_1, \zeta''_1 = 0$ . In particular we see from  $\phi(U) \subset f(\mathbb{D}_{r'}(\xi_0)) \subset f(\mathbb{D}_r(0)) \subset \mathcal{E}$  that

$$w := \phi_n(a_\nu + \xi) \in \mathcal{E}_{z+(\zeta+\zeta')}, \quad w' := \phi_n(a_\nu - \xi) \in \mathcal{E}_{z-(\zeta+\zeta'')}.$$

Observe that  $\zeta, \zeta', \zeta''$  satisfy the conditions in (9). Since  $a_\nu \in U$  and  $\phi'_1 \equiv 1$  on  $U$  we get  $\|\zeta' + 2\zeta + \zeta''\| \geq (2 - C)\|\zeta\| = (2 - C)\|D\phi_*(a_\nu)(\xi)\| \geq (2 - C)|\xi|$  and thus, in view of Lemma 8, can finally make the following estimate:

$$\frac{|\phi_n(a_\nu + \xi) - \phi_n(a_\nu - \xi)|}{2|\xi|} \geq (1 - C/2) \cdot \frac{|w - w'|}{\|\zeta' + 2\zeta + \zeta''\|} \geq \frac{1 - C/2}{1 + C/2} \cdot (\nu - 1).$$

This holds true for every member  $\nu$  of the strictly increasing sequence  $(\nu_k)$  and the right term becomes unbounded as  $\nu \rightarrow +\infty$ . Since for each fixed  $\nu$  the number  $\xi$  was chosen such that  $a_\nu \pm \xi \in U$ , this contradicts the fact that  $\phi_n$  has a bounded derivate on  $U$ .  $\square$

## 4. Choice of the sequence $\{\varepsilon_l\}$ - Part II

Recall that  $E_\nu = \{P_\nu = 0\}$ ,  $\nu \in \mathbb{N}$ . We show that for  $\{\varepsilon_l\}$  decreasing fast enough we can guarantee nice convergence properties of the sequence  $\{P_\nu\}$  as well as certain relations between the limit set  $\mathcal{E}$  of  $\{E_\nu\}$  and the sublevel sets of the defining polynomials  $P_\nu$ .

**Lemma 10.** *Let  $\{\varepsilon_l\}$  be chosen in such a way that  $\varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^l$  on  $B^{n-1}(0, l)$  for every  $l \in \mathbb{N}$ . Then the sequence  $\{|P_\nu|^{1/2^\nu}\}$  converges uniformly on compact subsets of  $\mathbb{C}^n \setminus \mathcal{E}$  and  $\lim_{\nu \rightarrow \infty} |P_\nu|^{1/2^\nu} > 0$  on  $\mathbb{C}^n \setminus \mathcal{E}$ .*



**Proof.** Fix  $(z_0, w_0) \in \mathbb{C}^n \setminus \mathcal{E}$  and choose  $R > 0$  such that  $(z_0, w_0) \in \Delta_R := B^{n-1}(0, R) \times \mathbb{C}$ . Since  $\mathcal{E}$  is closed and  $E_\nu \cap \overline{\Delta_R} \rightarrow \mathcal{E} \cap \overline{\Delta_R}$  in the Hausdorff metric, there exist a ball  $B := B^n((z_0, w_0), \delta) \subset \Delta_R$  and positive numbers  $r > 0$ ,  $N_r > 0$  such that  $\text{dist}(B, E_\nu) > r$  for all  $\nu \geq N_r$ . Now for every  $\nu, \mu \in \mathbb{N}$ ,  $j \in \mathbb{N}_{2^\nu}$  and  $z \in \mathbb{C}^{n-1}$  we denote the  $2^\mu$  values of  $w_j^{(\nu)}(z) + \sum_{l=\nu+1}^{\nu+\mu} \varepsilon_l \sqrt{z_{[l]} - a_l}$  by  $w_1^{(\mu)}(\nu, j; z), \dots, w_{2^\mu}^{(\mu)}(\nu, j; z)$ . Observe that with this notation we have

$$|P_{\nu+\mu}(z, w)|^{1/2^{\nu+\mu}} = \prod_{l=1}^{2^{\nu+\mu}} |w - w_l^{(\nu+\mu)}(z)|^{1/2^{\nu+\mu}} = \prod_{j=1}^{2^\nu} \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(\nu, j; z)|^{1/2^{\nu+\mu}},$$

thus passing from  $|P_\nu(z, w)|^{1/2^\nu}$  to  $|P_{\nu+\mu}(z, w)|^{1/2^{\nu+\mu}}$  amounts to replace each term  $|w - w_j^{(\nu)}(z)|$  occurring in the product expansion of  $|P_\nu(z, w)|^{1/2^\nu}$  by the mean value  $\prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(\nu, j; z)|^{1/2^\mu}$ . Since for  $\nu \geq R$  one has  $|w_j^{(\nu)}(z) - w_k^{(\mu)}(\nu, j; z)| \leq \sum_{l=\nu+1}^{\nu+\mu} \varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^\nu$  for all  $z \in B^{n-1}(0, R)$ , we can estimate the resulting error by means of

$$\begin{aligned} \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(\nu, j; z)|^{1/2^\mu} &> \prod_{k=1}^{2^\mu} (|w - w_j^{(\nu)}(z)| - 1/2^\nu)^{1/2^\mu} = |w - w_j^{(\nu)}(z)| - 1/2^\nu \\ \prod_{k=1}^{2^\mu} |w - w_k^{(\mu)}(\nu, j; z)|^{1/2^\mu} &< \prod_{k=1}^{2^\mu} (|w - w_j^{(\nu)}(z)| + 1/2^\nu)^{1/2^\mu} = |w - w_j^{(\nu)}(z)| + 1/2^\nu \end{aligned}$$

to be less than  $1/2^\nu$  for all  $(z, w) \in B \subset B^{n-1}(0, R) \times \mathbb{C}$  (obviously the first inequality is trivial if  $|w - w_j^{(\nu)}(z)| < 1/2^\nu$ ). In particular whenever  $|w - w_j^{(\nu)}(z)| \geq 1/2^\nu$  on  $B$  and  $\nu \geq R$  we get

$$\prod_{j=1}^{2^\nu} (|w - w_j^{(\nu)}(z)| - 1/2^\nu)^{1/2^\nu} \leq |P_{\nu+\mu}(z, w)|^{1/2^{\nu+\mu}} \leq \prod_{j=1}^{2^\nu} (|w - w_j^{(\nu)}(z)| + 1/2^\nu)^{1/2^\nu}$$

on  $B$ . But  $|w - w_j^{(\nu)}(z)| > r$  on  $B$  for all  $\nu \geq N_r$  where  $r$  does not depend on  $\nu$ . Since  $|P_\nu(z, w)| = \prod_{j=1}^{2^\nu} |w - w_j^{(\nu)}(z)|$ , this shows that  $\{|P_\nu(z, w)|^{1/2^\nu}\}_{\nu \geq 1}$  is a Cauchy sequence for every  $(z, w) \in B$  and in fact that  $\{|P_\nu|^{1/2^\nu}\}_{\nu \geq 1}$  converges uniformly on  $B$ . Moreover  $\lim_{\nu \rightarrow \infty} |P_\nu|^{1/2^\nu} > 0$  on  $B$ , since the above estimates hold true for all  $\mu \in \mathbb{N}$ .  $\square$

**Lemma 11.** *If  $\{\varepsilon_l\}$  is decreasing fast enough, then*

$$\mathcal{E} = \bigcap_{\nu \in \mathbb{N}} \bigcup_{\mu \geq \nu} \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}. \quad (10)$$

Moreover the following relations hold true for every  $\mu \geq \nu \geq R$ :

1.  $\{|P_\mu| < (\frac{1}{\nu+1})^{2^\mu}\} \cap \overline{B^n}(0, R) \subset \subset \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}.$
2.  $\{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\} \cap \overline{B^n}(0, R) \subset \subset \{|P_\mu| < (\frac{1}{\nu-1})^{2^\mu}\}.$

**Proof.** For  $M \subset \mathbb{C}^n$  and  $R, \delta > 0$  we let  $M_{(R)} := M \cap \overline{B}^n(0, R)$  and

$$M^{(\delta)} := M \cup \bigcup_{x \in \partial M} B^n(x, \delta) \quad \text{and} \quad M^{(-\delta)} := M \setminus \bigcup_{x \in \partial M} B^n(x, \delta).$$

One easily verifies the following relations for all  $M, N \subset \mathbb{C}^n$  and  $R, \delta, \delta_1, \delta_2 > 0$ :

- A)  $M \subset N \Rightarrow M^{(\delta)} \subset N^{(\delta)}$  and  $M \subset N \Rightarrow M^{(-\delta)} \subset N^{(-\delta)}$ .
- B)  $M \subset N \Rightarrow M \subset\subset N^{(\delta)}$  and  $M \subset N \Rightarrow M^{(-\delta)} \subset\subset N$ , if  $M$  is bounded.
- C)  $[M^{(\delta_1)}]^{(\delta_2)} = M^{(\delta_1 + \delta_2)}$  and  $[M^{(-\delta_1)}]^{(-\delta_2)} = M^{(-(\delta_1 + \delta_2))}$ .
- D)  $[M^{(\delta)}]_{(R-\delta)} \subset [M_{(R)}]^{(\delta)}$  and  $[M^{(-\delta)}]_{(R-\delta)} \subset [M_{(R)}]^{(-\delta)}$ .

Moreover  $M_{(R)}^{(\pm\delta)}$  will denote the set  $M^{(\pm\delta)} \cap \overline{B}^n(0, R)$ . We can choose sequences  $\{\varepsilon_l\}, \{\delta_l\}$  of positive numbers converging to zero such that for all  $\nu \in \mathbb{N}$  the following relations hold true:

- (1 <sub>$\nu$</sub> )  $\varepsilon_\nu \sqrt{|z_{[\nu]} - a_\nu|} < \frac{1}{2^\nu}$  on  $B^{n-1}(0, \nu)$ .
- (2 <sub>$\nu$</sub> )  $\left[ \left\{ |P_\nu| < \left(\frac{1}{\nu+1}\right)^{2^\nu} \right\}_{(\nu+1)} \cup \left\{ |P_\nu| > \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(\nu+1)} \right] \cap \left\{ |P_\nu| = \left(\frac{1}{\nu}\right)^{2^\nu} \right\}^{(\delta_\nu)} = \emptyset$ .
- (3 <sub>$\nu+1$</sub> )  $\left\{ |P_{\nu+1}| < \left(\frac{1}{\lambda}\right)^{2^{\nu+1}} \right\}_{(\nu+1)} \subset \left\{ |P_\nu| < \left(\frac{1}{\lambda}\right)^{2^\nu} \right\}^{(\delta_\nu/2^\nu)}$  for  $\lambda = 1, \dots, \nu+1$ .
- (3' <sub>$\nu+1$</sub> )  $\left\{ |P_\nu| < \left(\frac{1}{\lambda}\right)^{2^\nu} \right\}_{(\nu+1)}^{(-\delta_\nu/2^\nu)} \subset \left\{ |P_{\nu+1}| < \left(\frac{1}{\lambda}\right)^{2^{\nu+1}} \right\}$  for  $\lambda = 1, \dots, \nu-1$ .

Indeed, we can choose  $\varepsilon_1$  to satisfy (1<sub>1</sub>). After fixing such  $\varepsilon_1$  the polynomial  $P_1$  is fixed and we can choose  $\delta_1 < 1/2$  to satisfy (2<sub>1</sub>). Suppose now that  $\varepsilon_l, \delta_l$  are already chosen for  $l = 1, 2, \dots, \nu$  such that (1 <sub>$\nu$</sub> )-(3' <sub>$\nu$</sub> ) hold true. By Lemma 1 we know that  $P_{\nu+1} \rightarrow P_\nu^2$  uniformly on compact subsets as  $\varepsilon_{\nu+1} \rightarrow 0$ , hence we can find  $\varepsilon > 0$  such that for  $\varepsilon_{\nu+1} < \varepsilon$  the polynomial  $P_{\nu+1}$  satisfies (3 <sub>$\nu+1$</sub> ) and (3' <sub>$\nu+1$</sub> ). Moreover we can find  $\varepsilon' > 0$  such that for  $\varepsilon_{\nu+1} < \varepsilon'$  the inequality (1 <sub>$\nu+1$</sub> ) holds true. We choose  $\varepsilon_{\nu+1} < \min\{\varepsilon, \varepsilon'\}$  and we point out that every  $\varepsilon'_{\nu+1} \in (0, \varepsilon_{\nu+1})$  would also be a proper choice for  $\varepsilon_{\nu+1}$ . For  $P_{\nu+1}$  now being fixed we can find  $\delta_{\nu+1} < \delta_\nu/2$  satisfying (2 <sub>$\nu+1$</sub> ).

(i) We now prove statement 1 of the lemma. In order to do this we need the following

CLAIM 1. For  $\mu > \nu \geq R$  one has

$$\left\{ |P_\mu| < \left(\frac{1}{\nu+1}\right)^{2^\mu} \right\}_{(R)} \subset \left\{ |P_\nu| < \left(\frac{1}{\nu+1}\right)^{2^\nu} \right\}^{(\sum_{l=\nu}^{\mu-1} \delta_l/2^l)}.$$

PROOF. Let  $\mu > \nu \geq R$  be fixed. For proving the statement of the claim we use reverse induction on  $\rho$  to show that

$$\left\{ |P_\mu| < \left(\frac{1}{\nu+1}\right)^{2^\mu} \right\}_{(R)} \subset \left\{ |P_\rho| < \left(\frac{1}{\nu+1}\right)^{2^\rho} \right\}^{(\sum_{l=\rho}^{\mu-1} \delta_l/2^l)} \quad \text{for } \rho = \mu-1, \dots, \nu. \quad (11)$$

The case  $\rho = \mu - 1$  follows immediately from  $(3_\mu)$  with  $\lambda = \nu + 1$ . Suppose that property (11) holds for some  $\rho \in \mathbb{N}$  such that  $\mu > \rho > \nu \geq R$ . Then one also has

$$\left\{ |P_\mu| < \left(\frac{1}{\nu+1}\right)^{2^\mu} \right\}_{(R)} \subset \left\{ |P_\rho| < \left(\frac{1}{\nu+1}\right)^{2^\rho} \right\}_{(R)}^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle}. \quad (12)$$

Hence applying  $(3_\rho)$  with  $\lambda = \nu + 1$  we can conclude that

$$\begin{aligned} & \left\{ |P_\rho| < \left(\frac{1}{\nu+1}\right)^{2^\rho} \right\}_{(\rho)} \subset \left\{ |P_{\rho-1}| < \left(\frac{1}{\nu+1}\right)^{2^{\rho-1}} \right\}^{\langle \delta_{\rho-1}/2^{\rho-1} \rangle} \\ \Rightarrow & \left[ \left\{ |P_\rho| < \left(\frac{1}{\nu+1}\right)^{2^\rho} \right\}_{(\rho)} \right]^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle} \subset \left[ \left\{ |P_{\rho-1}| < \left(\frac{1}{\nu+1}\right)^{2^{\rho-1}} \right\}^{\langle \delta_{\rho-1}/2^{\rho-1} \rangle} \right]^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle} \\ \Rightarrow & \left[ \left\{ |P_\rho| < \left(\frac{1}{\nu+1}\right)^{2^\rho} \right\}^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle} \right]_{(\rho - \sum_{l=\rho}^{\mu-1} \delta_l/2^l)} \subset \left\{ |P_{\rho-1}| < \left(\frac{1}{\nu+1}\right)^{2^{\rho-1}} \right\}^{\langle \sum_{l=\rho-1}^{\mu-1} \delta_l/2^l \rangle} \\ \Rightarrow & \left[ \left\{ |P_\rho| < \left(\frac{1}{\nu+1}\right)^{2^\rho} \right\}^{\langle \sum_{l=\rho}^{\mu-1} \delta_l/2^l \rangle} \right]_{(R)} \subset \left\{ |P_{\rho-1}| < \left(\frac{1}{\nu+1}\right)^{2^{\rho-1}} \right\}^{\langle \sum_{l=\rho-1}^{\mu-1} \delta_l/2^l \rangle}. \end{aligned}$$

This together with (12) completes our argument by induction and proves Claim 1.  $\square$

Observe that, since  $\{\delta_l\}$  is monotonically decreasing, we get from Claim 1 and B) the following property:

$$\left\{ |P_\mu| < \left(\frac{1}{\nu+1}\right)^{2^\mu} \right\}_{(R)} \subset \left\{ |P_\nu| < \left(\frac{1}{\nu+1}\right)^{2^\nu} \right\}^{\langle \delta_\nu \rangle}. \quad (13)$$

Fix now some  $\nu \geq R$ . We are going to show that

$$\left\{ |P_\nu| < \left(\frac{1}{\nu+1}\right)^{2^\nu} \right\}_{(R)}^{\langle \delta_\nu \rangle} \subset \left\{ |P_\nu| < \left(\frac{1}{\nu}\right)^{2^\nu} \right\}. \quad (14)$$

Note that (13) and (14) together prove 1. By definition we have

$$\left\{ |P_\nu| < \left(\frac{1}{\nu+1}\right)^{2^\nu} \right\}_{(R)}^{\langle \delta_\nu \rangle} = \left\{ |P_\nu| < \left(\frac{1}{\nu+1}\right)^{2^\nu} \right\}_{(R)} \cup \bigcup_{x \in \partial\{|P_\nu| < (\frac{1}{\nu+1})^{2^\nu}\}} B^n(x, \delta_\nu)_{(R)}$$

Obviously

$$\left\{ |P_\nu| < \left(\frac{1}{\nu+1}\right)^{2^\nu} \right\}_{(R)} \subset \left\{ |P_\nu| < \left(\frac{1}{\nu}\right)^{2^\nu} \right\}.$$

Let  $\zeta \in B^n(x, \delta_\nu)_{(R)}$  for some  $x \in \partial\{|P_\nu| < (\frac{1}{\nu+1})^{2^\nu}\}$ . Then in particular  $x \in \{|P_\nu| = (\frac{1}{\nu+1})^{2^\nu}\}_{(\nu+1)}$ . Assume to get a contradiction that  $\zeta \in \{|P_\nu| \geq (\frac{1}{\nu})^{2^\nu}\}$ . Since  $x \in \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}_{(\nu+1)}$  we then can find  $t \in (0, 1]$  such that  $\tilde{x} := (1 - t)x + t\zeta \in \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}$ . Now obviously  $\|\tilde{x} - x\| < \delta_\nu$  which shows that  $x \in \{|P_\nu| = (\frac{1}{\nu+1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle \delta_\nu \rangle}$ . In particular we conclude that  $\{|P_\nu| < (\frac{1}{\nu+1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle \delta_\nu \rangle} \neq \emptyset$  which contradicts  $(2_\nu)$ . This proves that

$$\bigcup_{x \in \partial\{|P_\nu| < (\frac{1}{\nu+1})^{2^\nu}\}} B^n(x, \delta_\nu)_{(R)} \subset \left\{ |P_\nu| < \left(\frac{1}{\nu}\right)^{2^\nu} \right\},$$

and hence (14). The proof of statement 1 of the lemma is now complete.

(ii) We now prove statement 2 of the lemma. For being able to do this we need the following

CLAIM 2. For  $\mu > \nu \geq R$  one has

$$\left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(R)}^{\langle -\sum_{l=\nu}^{\mu-1} \delta_l/2^l \rangle} \subset \left\{ |P_\mu| < \left(\frac{1}{\nu-1}\right)^{2^\mu} \right\}. \quad (15)$$

PROOF. Let  $\mu > \nu \geq R$  be fixed. For proving the statement of the claim we use induction on  $\rho$  to show that

$$\left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)}^{\langle -\sum_{l=\nu}^{\rho-1} \delta_l/2^l \rangle} \subset \left\{ |P_\rho| < \left(\frac{1}{\nu-1}\right)^{2^\rho} \right\}, \quad \text{for } \rho = \nu + 1, \dots, \mu. \quad (16)$$

The case  $\rho = \nu + 1$  follows immediately from  $(3'_{\nu+1})$  with  $\lambda = \nu - 1$ . Suppose that property (16) holds for some  $\rho \in \mathbb{N}$  such that  $\mu > \rho > \nu \geq R$ . Then we also have

$$\begin{aligned} & \left[ \left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)}^{\langle -\sum_{l=\nu}^{\rho-1} \delta_l/2^l \rangle} \right]_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)} \subset \left\{ |P_\rho| < \left(\frac{1}{\nu-1}\right)^{2^\rho} \right\} \\ \Rightarrow & \left[ \left[ \left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)}^{\langle -\sum_{l=\nu}^{\rho-1} \delta_l/2^l \rangle} \right]_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)} \right]^{\langle -\delta_\rho/2^\rho \rangle} \subset \left\{ |P_\rho| < \left(\frac{1}{\nu-1}\right)^{2^\rho} \right\}_{(\nu+1)}^{\langle -\delta_\rho/2^\rho \rangle} \\ \Rightarrow & \left[ \left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)}^{\langle -\sum_{l=\nu}^{\rho-1} \delta_l/2^l \rangle} \right]_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)} \subset \left\{ |P_\rho| < \left(\frac{1}{\nu-1}\right)^{2^\rho} \right\}_{(\nu+1)}^{\langle -\delta_\rho/2^\rho \rangle} \\ \Rightarrow & \left[ \left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)}^{\langle -\sum_{l=\nu}^{\rho-1} \delta_l/2^l \rangle} \right]_{(\nu+1-\sum_{l=\nu}^{\rho-1} \delta_l/2^l)} \subset \left\{ |P_\rho| < \left(\frac{1}{\nu-1}\right)^{2^\rho} \right\}_{(\nu+1)}^{\langle -\delta_\rho/2^\rho \rangle} \end{aligned}$$

while from  $(3'_{\rho+1})$  with  $\lambda = \nu - 1$  we get

$$\left\{ |P_\rho| < \left(\frac{1}{\nu-1}\right)^{2^\rho} \right\}_{(\nu+1)}^{\langle -\delta_\rho/2^\rho \rangle} \subset \left\{ |P_{\rho+1}| < \left(\frac{1}{\nu-1}\right)^{2^{\rho+1}} \right\}.$$

This completes our argument by induction and, since  $\nu + 1 - \sum_{l=\nu}^{\mu-1} \delta_l/2^l > R$ , proves Claim 2.  $\square$

Observe that, since  $\{\delta_l\}$  is monotonically decreasing, we get from Claim 2 and B) the following property:

$$\left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(R)}^{\langle -\delta_\nu \rangle} \subset \left\{ |P_\mu| < \left(\frac{1}{\nu-1}\right)^{2^\mu} \right\}. \quad (17)$$

Fix now some  $\nu \geq R$ . We are going to show that

$$\left\{ |P_\nu| < \left(\frac{1}{\nu}\right)^{2^\nu} \right\}_{(R)} \subset \left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(R)}^{\langle -\delta_\nu \rangle}. \quad (18)$$

Note that (17) and (18) together prove 2. By definition we have

$$\left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(R)}^{\langle -\delta_\nu \rangle} = \left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(R)} \setminus \bigcup_{x \in \partial\{|P_\nu| < (\frac{1}{\nu-1})^{2^\nu}\}} B^n(x, \delta_\nu)_{(R)}.$$

Obviously

$$\left\{ |P_\nu| < \left(\frac{1}{\nu}\right)^{2^\nu} \right\}_{(R)} \subset \left\{ |P_\nu| < \left(\frac{1}{\nu-1}\right)^{2^\nu} \right\}_{(R)}.$$

Let  $\zeta \in B^n(x, \delta_\nu)_{(R)}$  for some  $x \in \partial\{|P_\nu| < (\frac{1}{\nu-1})^{2^\nu}\}$ . Then in particular  $x \in \{|P_\nu| = (\frac{1}{\nu-1})^{2^\nu}\}_{(\nu+1)}$ . To get a contradiction assume that  $\zeta \in \{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}_{(R)}$ . Since  $x \in \{|P_\nu| > (\frac{1}{\nu})^{2^\nu}\}$  we then can find  $t \in (0, 1)$  such that  $\tilde{x} := (1-t)x + t\zeta \in \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}$ . Now obviously  $\|\tilde{x} - x\| < \delta_\nu$  which shows that  $x \in \{|P_\nu| = (\frac{1}{\nu-1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle \delta_\nu \rangle}$ . In particular we conclude that  $\{|P_\nu| > (\frac{1}{\nu-1})^{2^\nu}\}_{(\nu+1)} \cap \{|P_\nu| = (\frac{1}{\nu})^{2^\nu}\}^{\langle \delta_\nu \rangle} \neq \emptyset$  which contradicts  $(2_\nu)$ . This proves that

$$\{|P_\nu| < (\frac{1}{\nu})^{2^\nu}\}_{(R)} \cap \bigcup_{x \in \partial\{|P_\nu| < (\frac{1}{\nu-1})^{2^\nu}\}} B^n(x, \delta_\nu)_{(R)} = \emptyset,$$

and hence (18). The proof of statement 2 of the lemma is now complete.

Finally we show that the representation (10) holds true. Let  $(z, w) \in \mathbb{C}^n$  and choose  $R > 0$  such that  $(z, w) \in B^n(0, R)$ . Assume that  $(z, w) \in \mathcal{E}$ . Let  $\mu \geq R$ . Applying 1 we get

$$(E_{\mu+l})_{(R)} \subset \{|P_{\mu+l}| < (\frac{1}{\mu+l})^{2^{\mu+l}}\}_{(R)} \subset \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$$

for all  $l \in \mathbb{N}$ . But since  $(1_\rho)$  holds true for all  $\rho \in \mathbb{N}$ , we can apply Lemma 2 to see that  $\mathcal{E}_{(R)} = \lim_{l \rightarrow \infty} (E_{\mu+l})_{(R)}$  in the Hausdorff metric. Hence  $\mathcal{E}_{(R)} \subset \{|P_{\mu+1}| \leq (\frac{1}{\mu+1})^{2^{\mu+1}}\}_{(R)} \subset \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$ . Since this holds true for all  $\mu \geq R$ , it follows  $(z, w) \in \bigcap_{\nu \in \mathbb{N}} \bigcup_{\mu \geq \nu} \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$ . Conversely assume that  $(z, w) \notin \mathcal{E}$ . Then by Lemma 10 the sequence  $\{|P_\nu(z, w)|^{1/2^\nu}\}$  is converging to a positive real number, hence there exist  $\delta > 0$  and  $\mu_0 \in \mathbb{N}$  such that  $|P_\mu(z, w)|^{1/2^\mu} > \delta$  for all  $\mu \geq \mu_0$ . In particular  $(z, w) \notin \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$  for  $\mu \geq \max\{\mu_0, 1/\delta\}$  which shows that  $(z, w) \notin \bigcap_{\nu \in \mathbb{N}} \bigcup_{\mu \geq \nu} \{|P_\mu| < (\frac{1}{\mu})^{2^\mu}\}$ .  $\square$

## 5. Proof of the theorems. Open questions

We now fix the sequence  $\{\varepsilon_l\}$  once and for all to be converging to zero so fast that the conclusions of Lemma 9 and 11 hold true and that  $\varepsilon_l \sqrt{|z_{[l]} - a_l|} < 1/2^l$  on  $B^{n-1}(0, l)$ .

For each  $\nu \in \mathbb{N}$  define a function  $\varphi_\nu: \mathbb{C}^n \rightarrow [-\infty, +\infty)$  as

$$\varphi_\nu(z, w) := \frac{1}{2^\nu} \log |P_\nu(z, w)|.$$

Then  $\varphi_\nu$  is a plurisubharmonic function in  $\mathbb{C}^n$ , pluriharmonic in  $\mathbb{C}^n \setminus E_\nu$ , and  $\varphi_\nu(z, w) = -\infty$  if and only if  $(z, w) \in E_\nu$ .

**Lemma 12.** *The sequence  $\{\varphi_\nu\}$  converges uniformly on compact subsets of  $\mathbb{C}^n \setminus \mathcal{E}$  to a pluriharmonic function  $\varphi: \mathbb{C}^n \setminus \mathcal{E} \rightarrow \mathbb{R}$  and  $\lim_{(z, w) \rightarrow (z_0, w_0)} \varphi(z, w) = -\infty$  for every  $(z_0, w_0) \in \mathcal{E}$ . In particular  $\varphi$  has an unique extension to a plurisubharmonic function on  $\mathbb{C}^n$ .*

**Proof.** Applying Lemma 10 we immediately see that  $\{\varphi_\nu\}$  converges uniformly on compact subsets of  $\mathbb{C}^n \setminus \mathcal{E}$ . In particular  $\varphi$  is pluriharmonic in  $\mathbb{C}^n \setminus \mathcal{E}$ . Let  $(z_0, w_0) \in \mathcal{E}$  and let  $\{(z_j, w_j)\}_{j \geq 1}$  be an arbitrary sequence of points converging to  $(z_0, w_0)$ . Let  $R \in \mathbb{N}$  be such that  $(z_0, w_0) \in B^n(0, R)$ . From part 1 of Lemma 11 we know that

$$\{|P_{\mu+1}| < (\frac{1}{\mu+1})^{2\mu+1}\} \cap \overline{B}^n(0, R) \subset \{|P_\mu| < (\frac{1}{\mu})^{2\mu}\}$$

for every  $\mu \geq R$ , thus it follows from

$$\mathcal{E} = \bigcap_{\nu \in \mathbb{N}} \bigcup_{\mu \geq \nu} \{|P_\mu| < (\frac{1}{\mu})^{2\mu}\}$$

that  $\mathcal{E} \cap \overline{B}^n(0, R) \subset \{|P_\nu| < (\frac{1}{\nu})^{2\nu}\}$  for all  $\nu \geq R$ . Hence for every  $\nu \geq R$  there exists  $j(\nu) \in \mathbb{N}$  such that  $(z_j, w_j) \in \{|P_\nu| < (\frac{1}{\nu})^{2\nu}\} \cap B^n(0, R)$  for all  $j \geq j(\nu)$ . But whenever  $(z_j, w_j) \in \{|P_\nu| < (\frac{1}{\nu})^{2\nu}\} \cap \overline{B}^n(0, R)$  we know from part 2 of Lemma 11 that also  $(z_j, w_j) \in \{|P_\mu| < (\frac{1}{\mu-1})^{2\mu}\}$  for each  $\mu \geq \nu$ . This means that  $\varphi_\mu(z_j, w_j) < -\log(\mu-1)$  for each  $\mu \geq \nu$ . Hence  $\varphi(z_j, w_j) \leq -\log(\mu-1)$  for each  $j \geq j(\nu)$ . This shows that  $\lim_{j \rightarrow \infty} \varphi(z_j, w_j) = -\infty$ .  $\square$

**Proof of Theorem 1.** By construction we have  $\mathcal{E} = \{z \in \mathbb{C}^n : \varphi(z) = -\infty\}$  and  $\varphi$  is pluriharmonic in  $\mathbb{C}^n \setminus \mathcal{E}$  by Lemma 12. Using the representation (10) of  $\mathcal{E}$  by sublevel sets of the polynomials  $P_\nu$  we get

$$\mathbb{C}^n \setminus \mathcal{E} = \bigcup_{\nu \in \mathbb{N}} \bigcap_{\mu \geq \nu} \{\varphi_\mu \geq -\log \mu\},$$

hence  $\mathbb{C}^n \setminus \mathcal{E}$  is pseudoconvex. It only remains to show that  $\widehat{\partial B^n(0, R)} \cap \mathcal{E} = \overline{B}^n(0, R) \cap \mathcal{E}$ . Using (10) and part 1 of Lemma 11 we see that for every  $(z, w) \in \mathbb{C}^n \setminus \mathcal{E}$  there exists  $\nu \in \mathbb{N}$  such that  $\overline{B}^n(0, R) \cap \mathcal{E} \subset \{|P_\nu| < (\frac{1}{\nu})^{2\nu}\}$  but  $|P_\nu(z, w)| \geq (\frac{1}{\nu})^{2\nu}$ , i.e.  $(z, w) \notin \widehat{\partial B^n(0, R)} \cap \mathcal{E}$ . But since clearly  $\widehat{\partial B^n(0, R)} \cap \mathcal{E} \subset \overline{B}^n(0, R)$  this shows that  $\widehat{\partial B^n(0, R)} \cap \mathcal{E} \subset \overline{B}^n(0, R) \cap \mathcal{E}$ . Concerning the other direction note that  $\widehat{\partial B^n(0, R)} \cap E_\nu = \overline{B}^n(0, R) \cap E_\nu$  for every  $\nu \in \mathbb{N}$  by the maximum modulus principle and the fact that  $E_\nu$  is the zero set of the polynomial  $P_\nu$ . Since on bounded subsets of  $\mathbb{C}^n$  the sequence  $\{E_\nu\}$  converges to  $\mathcal{E}$  in the Hausdorff metric, we thus conclude that  $\overline{B}^n(0, R) \cap \mathcal{E} = \lim_{\nu \rightarrow \infty} \overline{B}^n(0, R) \cap E_\nu = \lim_{\nu \rightarrow \infty} \widehat{\partial B^n(0, R)} \cap E_\nu \subset \widehat{\partial B^n(0, R)} \cap \mathcal{E}$ .  $\square$

**Proof of Theorem 2.** For each  $C_1 \in \mathbb{R}$  we define  $\Omega_{C_1} \subset \mathbb{C}^n$  to be the domain

$$\Omega_{C_1} := \{(z, w) \in \mathbb{C}^n : \varphi(z, w) + (\|z\|^2 + |w|^2) < C_1\},$$

where  $\varphi(z, w)$  is the function constructed in Lemma 12. It follows from the plurisubharmonicity of  $\varphi$  on  $\mathbb{C}^n$  that  $\Omega_{C_1}$  is strictly pseudoconvex. Obviously one also has that  $\mathcal{E} = \{\varphi = -\infty\} \subset \Omega_{C_1}$ . Further, by Sard's theorem, we can choose a constant  $C_1$  such that  $\Omega_{C_1}$  has  $C^\infty$ -smooth boundary. We fix such a constant  $C_1$  and define  $\Omega$  to be the domain  $\Omega_{C_1}$ . By construction  $\mathcal{E}$  contains no analytic variety of positive

dimension. Using the representation (10) of  $\mathcal{E}$  by sublevel sets of the polynomials  $P_\nu$  we get

$$\Omega \setminus \mathcal{E} = \bigcup_{\nu \in \mathbb{N}} \bigcap_{\mu \geq \nu} \left( \Omega \cap \{ \varphi_\mu \geq -\log \mu \} \right).$$

In particular  $\Omega \setminus \mathcal{E}$  is pseudoconvex and hence the projection  $\pi_n(E(\partial\Omega))$  of the envelope of holomorphy  $E(\partial\Omega)$  of  $\partial\Omega$  onto  $\mathbb{C}^n$  is contained in  $\overline{\Omega} \setminus \mathcal{E}$ .

It remains to show that  $E(\partial\Omega)$  is single-sheeted and coincides with  $\overline{\Omega} \setminus \mathcal{E}$ . Observe that for every  $a \in \mathbb{R}$  the set  $\overline{\Omega} \cap \{ \varphi \geq a \}$  is compact and hence, since  $\varphi_\nu \rightarrow \varphi$  uniformly on compact subsets of  $\mathbb{C}^n \setminus \mathcal{E}$ , for each  $a \in \mathbb{R}$  we can choose a natural number  $N(a) \in \mathbb{N}$  such that

$$\Omega \cap \{ \varphi > a \} \subset \Omega \cap \{ \varphi_{N(a)} > a - 1 \} = \Omega \cap \{ |P_{N(a)}| > e^{2^{N(a)}(a-1)} \} \subset \Omega \cap \{ \varphi > a - 2 \}.$$

Fix some  $a \in \mathbb{R}$  and let  $N := N(a)$ . Observe that  $P_N$ , being a polynomial, has only finitely many singular values  $c_1, c_2, \dots, c_k$  and let  $S_N := \bigcup_{j=1}^k \{ P_N = c_j \}$  (indeed, using the explicit formula for  $P_N$  stated after Lemma 1, one can even see that  $k = 1$  and  $c_1 = 0$ ). Let now  $f \in CR(\partial\Omega)$ . Since  $\Omega$  is strictly pseudoconvex,  $f$  extends to a holomorphic function on some one-sided neighbourhood  $U \subset \overline{\Omega}$  of  $\partial\Omega$ , which will be denoted by  $f$  as well.

Let  $H \subset \mathbb{C}^n$  denote a complex two-dimensional affine subspace of  $\mathbb{C}^n$  which is obtained by fixing  $n - 2$  of the coordinates  $z_1, z_2, \dots, z_{n-1}, w$  (for  $n = 2$  the only possible choice is  $H = \mathbb{C}^2$ ). Then  $\Omega \cap H = \bigcup_\alpha \Gamma_\alpha$  is the disjoint union of a family  $\{\Gamma_\alpha\}$  of strictly pseudoconvex domains  $\Gamma_\alpha \subset H \cong \mathbb{C}^2$ , and  $\partial_H \Gamma_\alpha \subset \partial\Omega \cap H$  for each  $\alpha$ , where  $\partial_H \Gamma_\alpha$  denotes the boundary of  $\Gamma_\alpha$  with respect to the relative topology on  $H$ . In particular we can view each  $\Gamma_\alpha$  as a strictly pseudoconvex domain in  $\mathbb{C}^2$  and for each  $\alpha$  the restriction of  $f$  to  $U \cap H$  defines a holomorphic function in a one-sided neighbourhood of  $\partial_H \Gamma_\alpha$ . With the situation reduced to a two-dimensional case we can now argue as in the example from introduction and conclude from Theorem A in [J] that  $E(\partial_H \Gamma_\alpha)$  is single-sheeted (of course here  $E(\partial_H \Gamma_\alpha)$  denotes the envelope of holomorphy of  $\partial_H \Gamma_\alpha$  with respect to functions holomorphic in  $H \cong \mathbb{C}^2$ ). On the other hand, since for each  $\nu \in \mathbb{N}$  the restriction  $P_\nu|_H$  is again a polynomial and we can assume it to be nonconstant (for  $\nu \geq \nu_0$  big enough this clearly is satisfied), for each  $a' \in \mathbb{R}$  the sets  $\{P_{N(a')} = c\}$  with  $c \in \mathbb{C}$ ,  $|c| > e^{2^{N(a')}(a'-1)}$ , constitute a continuous family of analytic curves in  $H \cong \mathbb{C}^2$  that fills  $(\Omega \cap H) \cap \{ \varphi > a' \}$ . Using the Kontinuitätssatz we thus conclude that  $E(\partial_H \Gamma_\alpha) = \overline{\Gamma}_\alpha \cap \{ \varphi > -\infty \} = \overline{\Gamma}_\alpha \setminus \mathcal{E}$  for each  $\alpha$ . Hence, since the domains  $\Gamma_\alpha$  are disjoint and pseudoconvex, we get that  $E(\bigcup_\alpha \partial_H \Gamma_\alpha)$  is single-sheeted and  $(\Omega \cap H) \setminus \mathcal{E} \subset E(\bigcup_\alpha \partial_H \Gamma_\alpha)$ . In particular  $f|_{U \cap H}$  extends to a holomorphic function

$$f_H: (\Omega \setminus \mathcal{E}) \cap H \rightarrow \mathbb{C}, \quad f_H = f \text{ near } \partial\Omega.$$

Observe that this already proves our claim in the case  $n = 2$ .

Assume now that  $n \geq 3$ . For each  $c \in \mathbb{C} \setminus \{c_1, c_2, \dots, c_k\}$  the hypersurface  $\{P_N = c\}$  is a Stein manifold of dimension at least 2, and if  $|c| > e^{2^N(a-1)}$ , then each

connected component of  $\Omega_c := \Omega \cap \{P_N = c\}$  is a bounded strictly pseudoconvex domain in  $\{P_N = c\}$ . Further  $f$  restricts to a holomorphic function on  $\Omega_c \setminus K$ , where  $K \subset \Omega_c$  is a compact set of the form  $K = \Omega_c \setminus \tilde{U}$  for some one-sided neighbourhood  $\tilde{U} \subset U$  of  $\partial\Omega$ . Since each connected component  $\Gamma$  of  $\Omega_c$  is bounded and strictly pseudoconvex, the boundary of  $\Gamma$  in  $\{P_N = c\}$  is connected and hence we can assume  $\Gamma \setminus K = \Gamma \cap \tilde{U}$  to be connected. Thus we can apply Hartogs theorem on removability of compact singularities to extend  $f|_{\Omega_c \setminus K}$  to a holomorphic function  $\tilde{f}_c$  on  $\Omega_c$  (for a version of the classical Hartogs theorem in the setting of Stein manifolds see [AH]). In this way we can define a function

$$f_a: [\Omega \cap \{P_N > e^{2^N(a-1)}\}] \setminus S \rightarrow \mathbb{C}, \quad f_a = f \text{ near } \partial\Omega,$$

by letting  $f_a(z, w) = \tilde{f}_c(z, w)$  if  $P_N(z, w) = c$ . We claim that for every two-dimensional subspace  $H \subset \mathbb{C}^n$  described above the functions  $f_a$  and  $f_H$  coincide on their common domain of definition, namely on the set  $[\Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}] \setminus S$ . Indeed, let  $c \in \mathbb{C} \setminus \{c_1, c_2, \dots, c_k\}$ ,  $|c| > e^{2^N(a-1)}$ . Since the restriction  $P_N|_H$  is again a (nonconstant) polynomial, the set  $\gamma_c := \Omega \cap H \cap \{P_N = c\}$  is an analytic curve in  $\Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}$ . Observe that the boundary of  $\gamma_c$  is contained in  $\partial\Omega$  and recall that  $f_a$  and  $f_H$  are holomorphic on  $\gamma_c$  and coincide near  $\partial\Omega$ . Thus it follows from the uniqueness theorem that  $f_a = f_H$  on  $\gamma_c$ . Hence, since  $c \in \mathbb{C} \setminus \{c_1, c_2, \dots, c_k\}$  with  $|c| > e^{2^N(a-1)}$  was arbitrary, we conclude that

$$f_a = f_H \quad \text{on} \quad [\Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}] \setminus S. \quad (19)$$

In particular this shows that  $f_a$  is holomorphic in each variable separately (recall the definition of  $H$ ). Thus by Hartogs separate analyticity theorem  $f_a$  is a holomorphic function on  $[\Omega \cap \{P_N > e^{2^N(a-1)}\}] \setminus S$ . Moreover we see from (19) and the holomorphicity of  $f_H$  on  $(\Omega \cap H) \setminus \mathcal{E} \supset \Omega \cap H \cap \{P_N > e^{2^N(a-1)}\}$  that  $f_a$  remains bounded near  $S$ . It follows then from Riemann's removable singularities theorem that  $f_a$  extends to a holomorphic function  $\tilde{f}_a$  on  $\Omega \cap \{P_N > e^{2^N(a-1)}\} \supset \Omega \cap \{\varphi > a\}$ . Since  $a \in \mathbb{R}$  was arbitrary, and since  $\Omega \setminus \mathcal{E} = \bigcup_{a \in \mathbb{R}} \Omega \cap \{\varphi > a\}$ , we conclude that  $f$  has a single-valued holomorphic extension to  $\overline{\Omega} \setminus \mathcal{E}$ . Hence  $E(\partial\Omega)$  is single-sheeted and  $E(\partial\Omega) = \overline{\Omega} \setminus \mathcal{E}$ .

Now we can construct a  $CR$  function  $f$  on  $\partial\Omega$  which extends exactly to  $\overline{\Omega} \setminus \mathcal{E}$ . In order to do so let

$$\tilde{\Omega} := \{(z, w) \in \mathbb{C}^n : \varphi(z, w) + (\|z\|^2 + |w|^2) < C_2\}$$

where the constant  $C_2 > C_1$ . Then the domain  $\tilde{\Omega}$  is also pseudoconvex and  $\overline{\Omega} \subset \tilde{\Omega}$ . As before we see that  $\tilde{\Omega} \setminus \mathcal{E}$  is pseudoconvex, hence there exists a holomorphic function  $f \in \mathcal{O}(\tilde{\Omega} \setminus \mathcal{E})$  which does not extend to  $\mathcal{E}$ . Then  $f|_{\partial\Omega}$  is a function as required.  $\square$

Finally we state some open questions related to the content of the paper (and also related to each other).



**Question 1.** Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be an unbounded strictly pseudoconvex domain. For each  $R > 0$  consider the hull  $\partial\Omega \cap \widehat{\overline{B}^n(0, R)}_{\mathcal{A}(\Omega)}$  of the set  $\partial\Omega \cap \overline{B}^n(0, R)$  with respect to the algebra  $\mathcal{A}(\Omega)$  of the functions holomorphic in  $\Omega$  which are continuous up to the boundary  $\partial\Omega$ . Is it true that  $\bigcup_{R>0} \widehat{\partial\Omega \cap \overline{B}^n(0, R)}_{\mathcal{A}(\Omega)} = \overline{\Omega}$ ?

**Question 2.** Is it true that there exist a properly embedded into  $\mathbb{C}^n$ ,  $n \geq 2$ , smooth Levi-flat hypersurface  $\mathcal{M}$  and an unbounded strictly pseudoconvex domain  $\Omega \subset \mathbb{C}^n$  such that  $\mathcal{M} \subset \Omega$ ?

**Question 3.** Let  $\Omega \subset \mathbb{C}^n$ ,  $n \geq 2$ , be an unbounded strictly pseudoconvex domain. Does it follow that its boundary  $\partial\Omega$  is connected?

**Remark.** After submitting this paper to arXiv the authors were informed by M. Brunella that the answers to Questions 1 and 3 are negative and to Question 2 is positive.

To show this M. Brunella suggests to consider a domain  $W \subset \mathbb{C}_{z,w}^2$  biholomorphic to  $\mathbb{C}_{z,w}^2$  such that  $\{z \in \mathbb{C}_z : (z, 0) \in W\} = \bigcup_{k=1}^N D_k$  and  $\{z \in \mathbb{C}_z : (z, 0) \in \overline{W}\} = \bigcup_{k=1}^N \overline{D}_k$ , where  $D_1, D_2, \dots, D_N$  are bounded domains in  $\mathbb{C}_z$  with  $C^1$ -smooth boundaries such that  $\overline{D}_1, \overline{D}_2, \dots, \overline{D}_N$  are pairwise disjoint. The existence of such a domain is granted by Corollary 1.1 in [G]. Let now  $\delta > 0$  be so small that the  $\delta$ -neighbourhoods  $U_k^\delta$  of  $D_k$  in  $\mathbb{C}_z$ ,  $k = 1, 2, \dots, N$ , are still pairwise disjoint and for each  $k = 1, 2, \dots, N$  consider a strictly subharmonic function  $\varphi_k^\delta \in C^\infty(\overline{U}_k^\delta)$  such that  $\partial[\overline{W} \cap \{(z, w) \in \mathbb{C}^2 : z \in U_k^\delta, |w| \leq e^{\varphi_k^\delta(z)}\}] \subset \{(z, w) \in \mathbb{C}^2 : z \in U_k^\delta, |w| = e^{\varphi_k^\delta(z)}\}$  and such that the set  $\{(z, w) \in \mathbb{C}^2 : z \in \partial U_k^\delta, |w| = e^{\varphi_k^\delta(z)}\}$  is disjoint from  $\overline{W}$ . Observe that these conditions are satisfied for  $\delta$  small enough and  $\varphi_k^\delta$  close enough to  $-\infty$  due to the above property of  $\overline{W}$ . Fix now such  $\delta$  and  $\varphi_k^\delta$ ,  $k = 1, 2, \dots, N$ , and consider an unbounded connected component  $\widetilde{W}$  of the set  $W \setminus \bigcup_{k=1}^N \{(z, w) \in \mathbb{C}^2 : z \in U_k^\delta, |w| \leq e^{\varphi_k^\delta(z)}\}$ . Then by construction  $\widetilde{W}$  is strictly pseudoconvex along  $\partial\widetilde{W} \setminus \partial W$  and, moreover,  $\partial\widetilde{W} \setminus \partial W$  has at least  $N$  different connected components. Let  $F$  be a biholomorphic map of  $W$  to  $\mathbb{C}^2$  and define the domain  $\Omega$  by  $\Omega := F(\widetilde{W})$ . Then, since  $\partial\Omega = F(\partial\widetilde{W} \setminus \partial W)$ ,  $\Omega$  is an unbounded strictly pseudoconvex domain in  $\mathbb{C}^2$  with at least  $N$  boundary components which gives a negative answer to Question 3. This domain contains a properly embedded into  $\mathbb{C}^2$  Levi-flat hypersurface, namely, the surface  $F((\partial D \times \mathbb{C}_w) \cap W)$ , where  $D$  is any open disc in  $\pi_z(W) \setminus \bigcup_{k=1}^N U_k^\delta$ . This gives a positive answer to Question 2. Finally let  $\Omega' \subset \mathbb{C}^2$  be a strictly pseudoconvex domain such that  $\overline{\Omega} \subset \Omega'$  and  $F^{-1}(\Omega') \cap \{w = 0\} = \emptyset$  (such a domain  $\Omega'$  can be obtained, for example, by repeating the construction of  $\Omega$  with  $\varphi_k^\delta$  replaced by  $\varphi_k^\delta - 1$ ,  $k = 1, 2, \dots, N$ ). Then  $\phi := (\log|w|) \circ F^{-1}$  is a continuous plurisubharmonic function on  $\Omega'$ , hence  $\Omega'_c := \{(z, w) \in \Omega' : \phi(z, w) < c\}$  is Runge in  $\Omega'$  for every  $c \in \mathbb{R}$  (see Corollary 1 of §4 in [N]). Moreover, by construction of  $\Omega$ , for suitably chosen  $c$  the set  $\Omega'_c$  is a neighbourhood of  $\partial\Omega$  and  $\overline{\Omega} \setminus \Omega'_c \neq \emptyset$ . After fixing such  $c$  we then conclude

that  $\bigcup_{R>0} \widehat{\partial\Omega \cap \overline{B^n}(0, R)}_{A(\Omega)} \subset \bigcup_{R>0} \widehat{\partial\Omega \cap \overline{B^n}(0, R)}_{\mathcal{O}(\Omega')} \cap \overline{\Omega} \subset \Omega'_c \cap \overline{\Omega} \subsetneq \overline{\Omega}$  which gives a negative answer to Question 1.

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